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Vector cascade algorithms and refinable function vectors in Sobolev spaces

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Abstract

In this paper we shall study vector cascade algorithms and refinable function vectors with a general isotropic dilation matrix in Sobolev spaces. By introducing the concept of a canonical mask for a given matrix mask and by investigating several properties of the initial function vectors in a vector cascade algorithm, we are able to take a relatively unified approach to study several questions such as convergence, rate of convergence and error estimate for a perturbed mask of a vector cascade algorithm in a Sobolev space $W_p^k(\mathbb{R}^s)$ $(1 \leqslant p \leqslant \infty,\ k \in \mathbb{N} \cup \{0\})$. We shall characterize the convergence of a vector cascade algorithm in a Sobolev space in various ways. As a consequence, a simple characterization for refinable Hermite interpolants and a sharp error estimate of a vector cascade algorithm in a Sobolev space with a perturbed mask will be presented. The approach in this paper enables us to answer some unsolved questions in the literature on vector cascade algorithms and to comprehensively generalize and improve results on scalar cascade algorithms and scalar refinable functions to the vector case.
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Keywords: Vector cascade algorithm and subdivision scheme; Refinable function vector; Hermite interpolant; Initial function vector; Canonical mask; Convergence; Rate of convergence; Error estimate; Smoothness

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1. Introduction

Refinable function vectors and vector subdivision schemes, as two of the most important and extensively studied fundamental objects in the literature of wavelet analysis, are useful in many applications such as signal processing and computer aided geometric design [3,9,12,21,23–25,34,37,45,46]. A vector cascade algorithm is closely related to a vector subdivision scheme. It is the purpose of this paper to study refinable function vectors and vector cascade algorithms in a relatively unified approach to have a better picture and understanding of some of their properties.

An $s\times s$ integer matrix M is called a *dilation matrix* if all its eigenvalues are greater than one in modulus. In this paper, we are concerned with the following vector refinement equation

$$\phi = |\det M| \sum_{\beta\in\mathbb{Z}^s} a(\beta)\phi(M\cdot-\beta), \tag{1.1}$$

where $\phi=(\phi_1,\dots,\phi_r)^T$ is called an *M-refinable function vector* which is an $r\times 1$ column vector of compactly supported functions or distributions, and a is called a (matrix) *mask* with multiplicity r which is a finitely supported sequence of $r\times r$ complex-valued matrices on $\mathbb{Z}^s$.

Let $\mathbb{N}_0$ denote all the nonnegative integers. For $\mu=(\mu_1,\dots,\mu_s)\in\mathbb{N}_0^s$, we denote $|\mu|:=|\mu_1|+\cdots+|\mu_s|$, $\mu!:=\mu_1!\cdots\mu_s!$ and $\xi^\mu:=\xi_1^{\mu_1}\cdots\xi_s^{\mu_s}$ for $\xi=(\xi_1,\dots,\xi_s)\in\mathbb{R}^s$. The partial derivative of a differentiable function f with respect to the jth coordinate is denoted by D_jf, $j=1,\dots,s$, and for $\mu=(\mu_1,\dots,\mu_s)\in\mathbb{N}_0^s$, D^μ is the differential operator $D_1^{\mu_1}\cdots D_s^{\mu_s}$. We denote by $W_p^k(\mathbb{R}^s)$ the *Sobolev space* that consists of all functions f such that $D^\mu f\in L_p(\mathbb{R}^s)$ for all $\mu\in\mathbb{N}_0^s$ and $|\mu|\leqslant k$, equipped with the norm defined by

$$||f||_{W_p^k(\mathbb{R}^s)} := \sum_{|\mu|\leqslant k} ||D^\mu f||_{L_p(\mathbb{R}^s)}.$$

For a Banach space $(B,||\cdot||_B)$, we denote $(B^{m\times n},||\cdot||_{B^{m\times n}})$ the Banach space of all $m\times n$ matrices $(b_{j,k})_{1\leqslant j\leqslant m,1\leqslant k\leqslant n}$ whose entries are elements in B, equipped with the following norm:

$$||(b_{j,k})_{1\leqslant j\leqslant m,1\leqslant k\leqslant n}||_{B^{m\times n}} := ||(||b_{j,k}||_B)_{1\leqslant j\leqslant m,1\leqslant k\leqslant n}||_{\mathbb{R}^{m\times n}},$$

where $||\cdot||_{\mathbb{R}^{m\times n}}$ denotes some norm on $\mathbb{R}^{m\times n}$. Note that all norms $||\cdot||_{\mathbb{R}^{m\times n}}$ on $\mathbb{R}^{m\times n}$ are equivalent. In particular, $\mathbb{R}^s:=\mathbb{R}^{s\times 1}$ for short.

Start with some appropriate initial function vector $\phi_0\in(W_p^k(\mathbb{R}^s))^{r\times 1}$. In order to solve the vector refinement equation (1.1), we employ the iteration scheme $Q_{a,M}^n\phi_0$ $(n\in\mathbb{N}_0)$, where $Q_{a,M}$ is the *cascade operator* on $(L_p(\mathbb{R}^s))^{r\times 1}$ $(1\leqslant p\leqslant\infty)$ given by

$$Q_{a,M}f := |\det M| \sum_{\beta\in\mathbb{Z}^s} a(\beta)f(M\cdot-\beta), \quad f\in(L_p(\mathbb{R}^s))^{r\times 1}. \tag{1.2}$$

This iteration scheme is called a (vector) *cascade algorithm* (see [3,9]) associated with mask a and dilation matrix M. If ϕ is a fixed point of $Q_{a,M}$ (that is, $Q_{a,M}\phi = \phi$), then ϕ must satisfy (1.1). When the multiplicity $r = 1$, a vector cascade algorithm and a refinable function vector are called a scalar cascade algorithm and a scalar refinable function, respectively.

Vector cascade algorithms and various properties of refinable function vectors have been extensively studied in literature [1–47]. See Section 4 for detailed discussion on recent developments on cascade algorithms. This paper is largely motivated by the work in Chen et al. [4] on convergence of vector cascade algorithms and by the work in [23] on refinable Hermite interpolants and their applications in computer-aided geometric design.

Though vector cascade algorithms and vector subdivision schemes have been relatively well studied in the literature, there are still several unanswered questions in this area and we feel that a relatively unified and self-contained approach is helpful to have a better picture and understanding of these and related topics.

For a compactly supported $r \times 1$ function vector f on $\mathbb{R}^s$, we say that the shifts of f are *stable* (see [30]) if $\text{span}\{\hat{f}(\xi + 2\pi\beta)\colon \beta \in \mathbb{Z}^s\} = \mathbb{C}^{r\times 1}$ for all $\xi \in \mathbb{R}^s$, where the Fourier transform $\hat{g}$ of $g \in L_1(\mathbb{R}^s)$ is defined to be $\hat{g}(\xi) = \int_{\mathbb{R}^s} g(t)e^{-it\cdot\xi}\,dt$, $\xi \in \mathbb{R}^s$ and can be naturally extended to tempered distributions.

In the following, let us mention some questions that motivate this work.

Q1: As in [4], let Y_k denote the set of all appropriate initial function vectors in a cascade algorithm. It was asked in Chen et al. [4] that "It would be interesting to know whether there always exists some $F = (f_1, \ldots, f_r)^T$ in Y_k such that the shifts of $f_1, \ldots, f_r$ are stable."

Q2: Suppose that Q1 is true and the cascade algorithm with such an initial function vector F converges in a Sobolev space. Will the cascade algorithm with every initial function vector in Y_k converge in the Sobolev space?

Q3: As an interesting family of refinable function vectors, refinable Hermite interpolants are of interest in computer-aided geometric design (see [12,17,23,37,45,46]). How to characterize a refinable Hermite interpolant in terms of its mask?

Q4: In many situations, truncation and perturbation of a mask are needed in applications. How will the perturbation of a matrix mask affect its vector cascade algorithm and its refinable function vector?

The structure of the paper is as follows. In Section 2, we shall introduce some auxiliary results which are of interest in their own right. Then we shall demonstrate that based on a simple observation which converts a given matrix mask into a canonical mask, vector cascade algorithms and refinable function vectors can be essentially investigated using the same techniques for the scalar case. At the end of Section 2, we shall study the structures of two very important subspaces in wavelet analysis.

In Section 3, we shall investigate necessary conditions for the initial function vectors in a cascade algorithm. The difficulty in Q1 partially lies in the fact that the set Y_k, which is described in [4], has a rather complicated structure. Our investigation leads to a very simple way of describing the set Y_k of all possible initial function vectors and consequently allows us to affirmatively answer Q1 (See Proposition 3.5). As in [18], we shall also investigate the mutual relations among the initial function vectors in a cascade algorithm. It turns out that such mutual relations are very useful in investigating many problems related to cascade algorithms.

In Section 4, we shall characterize the convergence of a vector cascade algorithm in a Sobolev space in terms of its mask in various ways. In particular, we shall give a positive answer to Q2 (See Theorem 4.3). It turns out that there is a very important quantity $v_p(a, M)$ defined in (4.3) in Section 4 which connects the convergence of vector cascade algorithms with the smoothness of refinable function vectors. More precisely, when M is isotropic and the shifts of a refinable function vector ϕ with mask a and dilation matrix M are stable, the quantity $v_p(a, M)$ is equal to the critical L_p smoothness exponent of ϕ. On the other hand, we shall show in Section 4 that a vector cascade algorithm associated with mask a and dilation matrix M for every initial function vector in Y_k converges in the Sobolev space $W_p^k(\mathbb{R}^s)$ if and only if $v_p(a, M) > k$. In the rest of Section 4, we shall also investigate the rate of convergence of a vector cascade algorithm (See Theorem 4.4).

In Section 5, we shall completely characterize a refinable Hermite interpolant in terms of its mask which settles Q3 (See Corollary 5.2). We show that a refinable function vector ϕ with mask a and dilation M is a Hermite interpolant of order r if and only if its mask a is a Hermite interpolatory mask of order r and $v_\infty(a, M) > r$.

In Section 6, we shall study how the perturbation of a mask will affect its vector cascade algorithm and its refinable function vector. We settle Q4 by obtaining a sharp error estimate for a vector cascade algorithm and a refinable function vector with a perturbed mask in Section 6 (See Theorem 6.2). The results in Section 6 are not trivial generalizations of the corresponding results in the scalar case since when $r > 1$ the set Y_k of initial function vectors indeed depends on the perturbed mask and therefore, is not invariant under perturbation.

Since the quantity $v_p(a, M)$ is very important, in Section 7, we shall discuss how to compute the particular quantity $v_2(a, M)$ by an efficient numerical algorithm in [28] (See Theorem 7.1). We shall also discuss how to compute $v_p(a, M)$ by factorizing the symbol of a univariate matrix mask a (See Proposition 7.2).

In this paper, we not only give alternative proofs for and improve some known results in the literature, but also obtain some new results on vector cascade algorithms and refinable function vectors. Our approach in this paper is relatively unified and may yield relatively simple proofs. The approach in this paper will be helpful for other problems related to vector cascade algorithms and refinable function vectors; it also enables us to have a better understanding of vector subdivision schemes in the geometric setting. Moreover, when $k = 0$, the Sobolev space $W_p^k(\mathbb{R}^s)$ is the $L_p(\mathbb{R}^s)$ space and we observe that all the results and proofs in this paper hold for a general (not necessarily isotropic) dilation matrix.

2. Auxiliary results, canonical masks and two subspaces

In this section, we shall introduce some auxiliary results and the concept of a canonical mask for a given matrix mask. Then we shall investigate the structure of two subspaces which play an important role in analyzing various properties of vector cascade algorithms and refinable function vectors.

For $k \in \mathbb{N}_0$, let O_k be the ordered set $\{\mu \in \mathbb{N}_0^s\colon |\mu| = k\}$ under the lexicographic order. That is, $\nu = (\nu_1, \ldots, \nu_s)$ is less than $\mu = (\mu_1, \ldots, \mu_s)$ in the lexicographic order if $|\nu| < |\mu|$ or $\nu_j = \mu_j$ for $j = 1, \ldots, i-1$ and $\nu_i < \mu_i$. By $\#O_k$ we denote the cardinality of the set O_k. For an $s \times s$ matrix N, $S(N, O_k)$ is defined to be the following $(\#O_k) \times (\#O_k)$ matrix [19] uniquely determined by

$$\frac{(Nx)^\mu}{\mu!} = \sum_{\nu \in O_k} S(N, O_k)_{\mu,\nu} \frac{x^\nu}{\nu!}, \quad \mu \in O_k. \tag{2.1}$$

It is obvious that $S(A, O_k)S(B, O_k) = S(AB, O_k)$. For matrices $A = (a_{i,j})_{1 \leqslant i \leqslant I, 1 \leqslant j \leqslant J}$ and $B = (b_{\ell,n})_{1 \leqslant \ell \leqslant L, 1 \leqslant n \leqslant N}$, the (right) *Kronecker product* $A \otimes B$ is defined to be $(a_{i,j}B)_{1 \leqslant i \leqslant I, 1 \leqslant j \leqslant J}$; its $((i-1)L + \ell, (j-1)N + n)$-entry is $a_{i,j}b_{\ell,n}$ and can be conveniently denoted by $[A \otimes B]_{i,j;\ell,n}$. It is well known that $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$, $C \otimes (A + B) = (C \otimes A) + (C \otimes B)$, $(A \otimes B)(C \otimes E) = (AC) \otimes (BE)$ and $(A \otimes B)^T = A^T \otimes B^T$.

The following result generalizes [18, Proposition 2.6] and is convenient to deal with derivatives in Sobolev spaces.

Proposition 2.1. *Let $D := [D_1, \ldots, D_s]$ be the row vector of differentiation operators. Denote the $1 \times s^k$ row vector of kth order differentiation operators by $\otimes^k D := D \otimes \cdots \otimes D$ with k copies of D, where $\otimes$ denotes the (right) Kronecker product. Let N be an $s \times s$ real-valued matrix. For any matrix f of functions in $C^k(\mathbb{R}^s)$ and for any matrices B and C of complex numbers such that the multiplication BfC is well defined, then*

$$[\otimes^k D] \otimes [Bf(N\cdot)C](\cdot) = B([\otimes^k D] \otimes f)(N\cdot)([\otimes^k N] \otimes C), \tag{2.2}$$

or equivalently,

$$\mathscr{D}^k \otimes [Bf(N\cdot)C](\cdot) = B(\mathscr{D}^k \otimes f)(N\cdot)(S(N, O_k) \otimes C), \tag{2.3}$$

where $\mathscr{D}^k := (D^\mu)_{\mu \in O_k}$ is a $1 \times (\#O_k)$ row vector of kth order differentiation operators and $S(N, O_k)$ is defined in (2.1).

Proof. Let $F = BfC$. As in [18], it is easy to check that $[D \otimes [F(N\cdot)]]_{1,i;j,\ell} = [(DN) \otimes F]_{1,i;j,\ell}(N\cdot)$. So, $D \otimes [F(N\cdot)] = [(DN) \otimes F](N\cdot)$. By induction, we have

$$\begin{aligned}[\otimes^k D] \otimes [Bf(N\cdot)C](\cdot) &= [\otimes^k (DN)] \otimes [BfC](N\cdot) = B[\otimes^k (DN) \otimes (fC)](N\cdot) \\ &= B([\otimes^k D] \otimes f)(N\cdot)([\otimes^k N] \otimes C).\end{aligned}$$

In order to prove (2.3), we define a $(\#O_k) \times s^k$ matrix H by $H_{\mu,j} := 1$, if $[\otimes^k D]_{1,j} = D^\mu$, and 0, otherwise, for $j = 1, \dots, s^k$ and $\mu \in O_k$. Similarly, define an $s^k \times (\#O_k)$ matrix G by $G_{j,\mu} := 1$, if $j = \min\{i\colon [\otimes^k D]_{1,i} = D^\mu\}$, and 0, otherwise. It is easy to verify that

$$HG = I_{\#O_k}, \quad \mathscr{D}^k = (\otimes^k D)G, \quad \otimes^k D = (\mathscr{D}^k)H$$

and

$$S(N, O_k) = H(\otimes^k N)G.$$

It follows from (2.2) and the above relations that (2.3) holds. $\square$

The following result will be needed later and is of interest in its own right.

Lemma 2.2. *Let M be an $s \times s$ matrix. Let A, B, C, E, F be given matrices of 2π-periodic trigonometric polynomials such that A, B, C, E are square matrices. Let X be an unknown $m \times n$ matrix of 2π-periodic trigonometric polynomials such that $A(\xi)X(M^T\xi)B(\xi) - C(\xi)X(\xi)E(\xi) - F(\xi)$ is well defined. Suppose that $X(0)$ satisfies $A(0)X(0)B(0) = C(0)X(0)E(0) + F(0)$. If $(\otimes^j M) \otimes B(0)^T \otimes A(0) - I_{s^j} \otimes E(0)^T \otimes C(0)$ (or equivalently, $S(M, O_j) \otimes B(0)^T \otimes A(0) - I_{\#O_j} \otimes E(0)^T \otimes C(0)$) is invertible for all $j = 1, \dots, k$, then the following system of linear equations given by*

$$D^\mu[A(\cdot)X(M^T\cdot)B(\cdot)](0) = D^\mu[C(\cdot)X(\cdot)E(\cdot)](0) + D^\mu F(0), \quad 0 < |\mu| \leqslant k \tag{2.4}$$

has a unique solution for $\{D^\mu X(0)\colon 0 < |\mu| \leqslant k\}$.

Proof. It is well known that $\mathrm{vec}(CXE) = (E^T \otimes C)\mathrm{vec}(X)$, where for $X = (X_{i,j})_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$,

$$\mathrm{vec}(X) := (X_{1,1}, \dots, X_{m,1}, X_{1,2}, \dots, X_{m,2}, \dots, X_{1,n}, \dots, X_{m,n})^T.$$

Rewrite (2.4) as

$$\begin{aligned} &D^\mu[(B(\cdot)^T \otimes A(\cdot))\mathrm{vec}(X(M^T\cdot))](0) \\ &\quad = D^\mu[(E(\cdot)^T \otimes C(\cdot))\mathrm{vec}(X(\cdot))](0) + D^\mu[\mathrm{vec}(F)](0). \end{aligned}$$

So, it suffices to prove the claim with $B = E = I$. Now (2.4) becomes

$$\begin{aligned} &[\otimes^j D] \otimes [A(0)X(M^T\cdot) - C(0)X(\cdot)](0) \\ &\quad = [\otimes^j D] \otimes [(C(\cdot) - C(0))X(\cdot) + F(\cdot) + (A(0) - A(\cdot))X(M^T\cdot)](0) \\ &\quad =: G_j \end{aligned}$$

for $j = 1, \dots, k$. By the Leibniz differentiation formula, we observe that G_j only involves $D^\mu X(0), |\mu| < j$. So, by Proposition 2.1, we have

$$A(0)([\otimes^j D] \otimes X)(0)([\otimes^j M^T] \otimes I_n) - C(0)([\otimes^j D] \otimes X)(0)I_{s^j + n} = G_j$$

for $j = 1, \ldots, k$. That is,

$$([\otimes^j M] \otimes I_n \otimes A(0) - I_{s^j+n} \otimes C(0))\mathrm{vec}(([\otimes^j D] \otimes X)(0)) = \mathrm{vec}(G_j)$$

for $j = 1, \ldots, k$. Since the matrix $(\otimes^j M) \otimes I_n \otimes A(0) - I_{s^j+n} \otimes C(0)$ is invertible for every $j = 1, \ldots, k$, we have

$$\mathrm{vec}(([\otimes^j D] \otimes X)(0)) = ([\otimes^j M] \otimes I_n \otimes A(0) - I_{s^j+n} \otimes C(0))^{-1}\mathrm{vec}(G_j).$$

The proof is completed by induction on $j = 1, \ldots, k$. $\square$

A matrix M is *isotropic* if M is similar to a diagonal matrix $\mathrm{diag}(\sigma_1, \ldots, \sigma_s)$ such that $|\sigma_1| = \cdots = |\sigma_s| = |\det M|^{1/s}$. An $s \times s$ matrix M is isotropic [18] if and only if there exists a norm $\|\cdot\|_M$ on $\mathbb{C}^{s\times 1}$ such that

$$\|Mx\|_M = |\det M|^{1/s}\|x\|_M \quad \forall x \in \mathbb{C}^{s\times 1}. \tag{2.5}$$

When M is isotropic, $\|\cdot\|_M$ denotes a norm on $\mathbb{C}^{s\times 1}$ such that (2.5) holds. For a matrix or an operator A, we denote $\rho(A) := \lim_{n\to\infty} \|A^n\|^{1/n}$ the spectral radius of A. When A is an $s \times s$ isotropic matrix, we have $\rho(A) = |\det A|^{1/s}$.

The *Fourier series* or *symbol* of a sequence a on $\mathbb{Z}^s$ is defined to be

$$\hat{a}(\xi) := \sum_{\beta \in \mathbb{Z}^s} a(\beta) e^{-i\beta\cdot\xi}, \quad \xi \in \mathbb{R}^s. \tag{2.6}$$

Throughout this paper, we denote $a_n (n \in \mathbb{N}_0)$ to be the sequence defined by

$$\hat{a}_n(\xi) := \prod_{j=1}^{n} \hat{a}((M^T)^{n-j}\xi) = \hat{a}((M^T)^{n-1}\xi)\cdots\hat{a}(M^T\xi)\hat{a}(\xi). \tag{2.7}$$

The sequence a_n is closely related to a vector subdivision scheme used in computer-aided geometric design and plays an important role in investigating vector cascade algorithms and refinable function vectors.

Lemma 2.3. *Let M be an $s \times s$ isotropic dilation matrix. Suppose that $f_n (n \in \mathbb{N})$ are function vectors in $(W_p^k(\mathbb{R}^s))^{r\times 1}$ such that the sequence f_n converges to f_∞ in the Sobolev space $(W_p^k(\mathbb{R}^s))^{r\times 1}$, $1 \leqslant p \leqslant \infty$, and when $p > 1$ we additionally assume that all f_n vanish outside a fixed compact set of $\mathbb{R}^s$. Then*

$$\lim_{n\to\infty} \rho(M)^{kn}\hat{f}_n((M^T)^n\xi) = \lim_{n\to\infty} D^\mu[\hat{f}_n((M^T)^n\cdot)](\xi) = 0 \quad \forall \xi \neq 0, \ |\mu| \leqslant k.$$

Proof. Since all f_n are supported on a compact set when $1 < p \leqslant \infty$, by Hölder inequality, it follows from the assumption $\lim_{n\to\infty} \|f_n - f_\infty\|_{(W_p^k(\mathbb{R}^s))^{r\times 1}} = 0$ that $\lim_{n\to\infty} \|f_n - f_\infty\|_{(W_1^k(\mathbb{R}^s))^{r\times 1}} = 0$.

Let $N := M^T$ and ξ be a fixed nonzero point in $\mathbb{R}^s$. Since $\widehat{D^\mu f_n}(\xi) = (i\xi)^\mu \hat{f}_n(\xi)$ and

$$\|D^\mu(\widehat{f_n - f_\infty})(N^n\xi)\| \leqslant \|D^\mu(f_n - f_\infty)\|_{(L_1(\mathbb{R}^s))^{r\times 1}} \leqslant \|f_n - f_\infty\|_{(W_1^k(\mathbb{R}^s))^{r\times 1}}$$

for $|\mu| \leqslant k$, we have

$$||(iN^n\xi)^\mu \hat{f}_n(N^n\xi)|| = ||\widehat{D^\mu f_n}(N^n\xi)|| \leqslant ||\widehat{D^\mu f_\infty}(N^n\xi)|| + ||f_n - f_\infty||_{(W_1^k(\mathbb{R}^s))^{r\times 1}}.$$

By the Riemann–Lebesgue lemma, we conclude that

$$\lim_{n\to\infty} ||(iN^n\xi)^\mu \hat{f}_n(N^n\xi)|| = \lim_{n\to\infty} ||\widehat{D^\mu f_\infty}(N^n\xi)|| + \lim_{n\to\infty} ||f_n - f_\infty||_{(W_1^k(\mathbb{R}^s))^{r\times 1}} = 0 \quad \forall |\mu| \leqslant k.$$

The claim follows directly from the above identity, Proposition 2.1 and the assumption that M is an isotropic dilation matrix. $\square$

We denote by $\ell_0(\mathbb{Z}^s)$ the linear space of all finitely supported sequences on $\mathbb{Z}^s$. Similarly, $\ell_p(\mathbb{Z}^s)$ denotes the linear space of all sequences v on $\mathbb{Z}^s$ such that $||v||_{\ell_p(\mathbb{Z}^s)} := (\sum_{\beta\in\mathbb{Z}^s} |v(\beta)|^p)^{1/p} < \infty$. When K is a compact subset of $\mathbb{Z}^s$, $\ell(K)$ denotes the linear space of all $v \in \ell_0(\mathbb{Z}^s)$ such that v vanishes outside K. By δ we denote the *Dirac sequence* on $\mathbb{Z}^s$ such that $\delta(0) = 1$ and $\delta(\beta) = 0$ for all $\beta \in \mathbb{Z}^s \backslash \{0\}$.

Let a be a matrix mask with multiplicity r. We say that a satisfies the *sum rules* of order $k+1$ with respect to the dilation matrix M if there exists a sequence $y \in (\ell_0(\mathbb{Z}^s))^{1\times r}$ such that $\hat{y}(0) \neq 0$,

$$D^\mu[\hat{y}(M^T\cdot)\hat{a}(\cdot)](0) = D^\mu \hat{y}(0) \quad \forall |\mu| \leqslant k, \mu \in \mathbb{N}_0^s \tag{2.8}$$

and

$$D^\mu[\hat{y}(M^T\cdot)\hat{a}(\cdot)](2\pi\beta) = 0 \quad \forall |\mu| \leqslant k, \beta \in (M^T)^{-1}\mathbb{Z}^s \backslash \mathbb{Z}^s. \tag{2.9}$$

The following result generalizes [22, Theorem 2.2] to any dimension and is quite useful in studying vector cascade algorithms and refinable function vectors.

Proposition 2.4. *Let $y \in (\ell_0(\mathbb{Z}^s))^{1\times r}$ such that $\hat{y}(0) \neq 0$. Then there exists $U_y \in (\ell_0(\mathbb{Z}^s))^{r\times r}$ such that* det $\hat{U}_y(\xi)$ *is a nonzero constant* (*that is, the sequence having symbol $\hat{U}_y(\xi)^{-1}$ is finitely supported*) *and*

$$\hat{\tilde{y}}(\xi) = [\hat{\tilde{y}}_1(\xi), \ldots, \hat{\tilde{y}}_r(\xi)] := \hat{y}(\xi)\hat{U}_y(\xi)$$

satisfies

$$\hat{\tilde{y}}_1(0) = 1, \quad D^\mu \hat{\tilde{y}}_j(0) = 0 \quad \forall j = 2, \ldots, r \quad \textit{and} \quad |\mu| \leqslant k.$$

Let a be a finitely supported matrix mask with multiplicity r and let ϕ satisfy $\hat{\phi}(M^T\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$. Define

$$\hat{\tilde{a}}(\xi) := \hat{U}_y(M^T\xi)^{-1}\hat{a}(\xi)\hat{U}_y(\xi) \quad \textit{and} \quad \hat{\tilde{\phi}}(\xi) := \hat{U}_y(\xi)^{-1}\hat{\phi}(\xi).$$

Then $\hat{\tilde{\phi}}(M^T\xi) = \hat{\tilde{a}}(\xi)\hat{\tilde{\phi}}(\xi)$. The equation in (2.8) *holds if and only if*

$$D^\mu[\hat{\tilde{y}}(M^T\cdot)\hat{\tilde{a}}(\cdot)](0) = D^\mu \hat{\tilde{y}}(0) \quad \textit{for all } |\mu| \leqslant k.$$

Therefore, when (2.8) *holds,* $\hat{\tilde{a}}(\xi)$ *must take the form*:

$$\begin{bmatrix} \hat{a}_{1,1}(\xi) & \hat{a}_{1,2}(\xi) \\ \hat{a}_{2,1}(\xi) & \hat{a}_{2,2}(\xi) \end{bmatrix} \quad \text{with } \hat{a}_{1,1}(0) = 1, \quad D^{\mu}\hat{a}_{1,2}(0) = 0 \quad \forall |\mu| \leqslant k, \tag{2.10}$$

where $a_{1,1} \in \ell_0(\mathbb{Z}^s)$, $a_{1,2} \in (\ell_0(\mathbb{Z}^s))^{1\times(r-1)}$, $a_{2,1} \in (\ell_0(\mathbb{Z}^s))^{(r-1)\times 1}$ *and* $a_{2,2} \in (\ell_0(\mathbb{Z}^s))^{(r-1)\times(r-1)}$. *Moreover, the following statements are equivalent*:

(a) *The mask* a *satisfies the sum rules of order* $k+1$ *in* (2.8) *and* (2.9) *with the sequence* y;
(b) *The mask* $\tilde{a}$ *satisfies the sum rules of order* $k+1$ *in* (2.8) *and* (2.9) *with the sequence* $\tilde{y}$ *whose symbol is* $[\hat{\tilde{y}}_1(\xi), 0, \ldots, 0]$;
(c) $\hat{\tilde{a}}(\xi)$ *takes the form of* (2.10) *and*

$$D^{\mu}\hat{a}_{1,1}(2\pi\beta) = 0 \quad \text{and} \quad D^{\mu}\hat{a}_{1,2}(2\pi\beta) = 0$$
$$\forall |\mu| \leqslant k, \ \beta \in (M^T)^{-1}\mathbb{Z}^s \backslash \mathbb{Z}^s. \tag{2.11}$$

Proof. Write $\hat{y}(\xi) = [\hat{y}_1(\xi), \ldots, \hat{y}_r(\xi)]$. Since $\hat{y}(0) \neq 0$, we can assume $\hat{y}_1(0) \neq 0$; otherwise we can permute the entries in $\hat{y}$. Since $\hat{y}_1(0) \neq 0$, it is easy to see that there exist $c_j \in \ell_0(\mathbb{Z}^s)$, $j = 2, \ldots, r$ such that $D^{\mu}[\hat{y}_j(\cdot) - \hat{c}_j(\cdot)\hat{y}_1(\cdot)](0) = 0$ for all $|\mu| \leqslant k$ and $j = 2, \ldots, r$, or equivalently, $D^{\mu}\hat{c}_j(0) = D^{\mu}[\hat{y}_j(\cdot)/\hat{y}_1(\cdot)](0)$ for all $|\mu| \leqslant k$ and $j = 2, \ldots, r$. Define $U_y \in (\ell_0(\mathbb{Z}^s))^{r\times r}$ by

$$\hat{U}_y(\xi) = \frac{1}{\hat{y}_1(0)} \begin{bmatrix} 1 & -\hat{c}(\xi) \\ 0 & I_{r-1} \end{bmatrix} \quad \text{with } \hat{c}(\xi) = [\hat{c}_2(\xi), \ldots, \hat{c}_r(\xi)].$$

It is easy to verify that $\hat{\tilde{y}}(\xi) = \hat{y}(\xi)\hat{U}_y(\xi)$ is desired. Other statements can be easily proved by a direct computation and by the Leibniz differentiation formula. □

We call a mask satisfying (2.10) and (2.11) a *canonical mask* of a given mask. The concept of a canonical mask allows us to investigate vector cascade algorithms and refinable function vectors using the same techniques for the scalar case. A canonical mask can preserve the symmetry of the original mask by appropriately choosing the matrix U_y [22].

The convolution of two sequences is defined to be

$$[u * v](\alpha) := \sum_{\beta \in \mathbb{Z}^s} u(\beta)v(\alpha - \beta), \quad u \in (\ell_0(\mathbb{Z}^s))^{\ell\times m}, \quad v \in (\ell_0(\mathbb{Z}^s))^{m\times n}.$$

Define a semi-convolution of a function and a sequence as follows:

$$u * f := \sum_{\beta \in \mathbb{Z}^s} u(\beta)f(\cdot - \beta), \quad u \in (\ell_0(\mathbb{Z}^s))^{\ell\times m}, \quad f \in (L_p(\mathbb{R}^s))^{m\times n}, \tag{2.12}$$

or $f * u := \sum_{\beta \in \mathbb{Z}^s} f(\cdot - \beta)u(\beta)$ for $f \in (L_p(\mathbb{R}^s))^{\ell \times m}$ and $u \in (\ell_0(\mathbb{Z}^s))^{m \times n}$. It is easy to verify that

$$u * (v * f) = (u * v) * f, \quad u \in (\ell_0(\mathbb{Z}^s))^{\ell \times m}, \quad v \in (\ell_0(\mathbb{Z}^s))^{m \times n}, \quad f \in (L_p(\mathbb{R}^s))^{n \times k}.$$

Given $y \in (\ell_0(\mathbb{Z}^s))^{1 \times r}$, we now define two interesting subspaces associated with y which play an important role in wavelet analysis. Let $D := [D_1, \dots, D_s]$ be the row vector of differentiation operators and let i denote the imaginary unit such that $i^2 = -1$. Observe that $(-iD)^\mu \hat{y}(0) = \sum_{\beta \in \mathbb{Z}^s} y(\beta)(-\beta)^\mu$ and

$$\begin{aligned}[p(\cdot - iD^T)\hat{y}](0) &:= \sum_{\mu \in \mathbb{N}_0^s} (D^\mu p)(\cdot) \frac{(-iD)^\mu}{\mu!} \hat{y}(0) \\ &= \sum_{\beta \in \mathbb{Z}^s} p(\cdot - \beta) y(\beta) = p * y, \quad p \in \Pi_k, \end{aligned} \tag{2.13}$$

where Π_k denotes the linear space of all polynomials with total degree no greater than k. Define

$$\mathscr{V}_{k,y} := \{v \in (\ell_0(\mathbb{Z}^s))^{r \times 1} : \, p * (y * v)(0) = 0 \quad \forall p \in \Pi_k\}. \tag{2.14}$$

By (2.13), we see that

$$\mathscr{V}_{k,y} = \{v \in (\ell_0(\mathbb{Z}^s))^{r \times 1} : \, D^\mu[\hat{y}(\cdot)\hat{v}(\cdot)](0) = 0 \quad \forall |\mu| \leqslant k\}.$$

Define

$$\begin{aligned}\mathscr{P}_{k,y} &:= \{p * y \in (\Pi_k)^{1 \times r} : \, p \in \Pi_k\} \\ &= \{[p(\cdot - iD^T)\hat{y}](0) \in (\Pi_k)^{1 \times r} : \, p \in \Pi_k\}. \end{aligned} \tag{2.15}$$

For $p \in (\Pi_k)^{m \times n}$, we shall use p to denote both the polynomial matrix $p(\cdot)$ and the polynomial sequence $(p(\beta))_{\beta \in \mathbb{Z}^s}$ since they can be easily distinguished in the context.

For a matrix A or an operator A acting on a finite-dimensional space V, we denote $\operatorname{spec}(A)$ or $\operatorname{spec}(A|_V)$ the multiset of all eigenvalues of A or $A|_V$ counting the multiplicity of the eigenvalues.

Proposition 2.5. *Let $y \in (\ell_0(\mathbb{Z}^s))^{1 \times r}$ such that $\hat{y}(0) \neq 0$. Let $\mathscr{V}_{k,y}$ and $\mathscr{P}_{k,y}$ be defined in* (2.14) *and* (2.15), *respectively. Then*

(1) $v \in \mathscr{V}_{k,y} \Rightarrow v(\cdot - \beta) \in \mathscr{V}_{k,y}$ *for all* $\beta \in \mathbb{Z}^s$; *that is,* $\mathscr{V}_{k,y}$ *is shift invariant*;
(2) $p \in \mathscr{P}_{k,y} \Rightarrow D^\mu p \in \mathscr{P}_{k,y}$ *and* $p(\cdot - \beta) \in \mathscr{P}_{k,y}$ *for all* $\mu \in \mathbb{N}_0^s$ *and* $\beta \in \mathbb{Z}^s$;
(3) $\mathscr{V}_{k,y} = \{v \in (\ell_0(\mathbb{Z}^s))^{r \times 1} : \, \sum_{\beta \in \mathbb{Z}^s} p(\beta) v(-\beta) = p * v(0) = 0 \; \forall p \in \mathscr{P}_{k,y}\}$;
(4) $\mathscr{P}_{k,y} = \{p \in (\Pi_k)^{1 \times r} : \, \sum_{\beta \in \mathbb{Z}^s} p(\beta) v(-\beta) = p * v(0) = 0 \quad \forall v \in \mathscr{V}_{k,y}\}$;

(5) *Let* U_y *be given in Proposition* 2.4. *Then* $\mathscr{V}_{k,y} = \operatorname{span}\{v(\cdot - \beta)\colon v \in \mathscr{B}_{k,y}, \beta \in \mathbb{Z}^s\}$, *that is*, $\mathscr{B}_{k,y}$ *generates the shift invariant space* $\mathscr{V}_{k,y}$, *where* $\mathscr{B}_{k,y}$ *is defined to be*

$$\mathscr{B}_{k,y} := \{v\colon \hat{v}(\xi) = \widehat{\nabla^\mu \delta}(\xi)\hat{U}_y(\xi)e_1,\ |\mu| = k+1\} \cup \{v\colon \hat{v}(\xi) = \hat{U}_y(\xi)e_j,\ j = 2, \dots, r\}, \tag{2.16}$$

where $\widehat{\nabla^\mu \delta}(\xi_1, \dots, \xi_s) := (1 - e^{-i\xi_1})^{\mu_1} \cdots (1 - e^{-i\xi_s})^{\mu_s}$ *for* $\mu = (\mu_1, \dots, \mu_s)$ *and* e_j *denotes the jth coordinate unit vector in* $\mathbb{R}^r$;

(6) *The mask a satisfies the sum rules of order* $k+1$ *in* (2.8) *and* (2.9) *with the sequence y if and only if* $S_{a,M}\mathscr{P}_{k,y} \subseteq \mathscr{P}_{k,y}$, *where the subdivision operator* $S_{a,M}$ *is defined to be*

$$S_{a,M}v(\alpha) := |\det M| \sum_{\beta \in \mathbb{Z}^s} v(\beta)a(\alpha - M\beta), \quad v \in (\ell_0(\mathbb{Z}^s))^{1\times r}; \tag{2.17}$$

(7) *The mask a satisfies the sum rules of order* $k+1$ *in* (2.8) *and* (2.9) *with the sequence y if and only if* $T_{a,M}\mathscr{V}_{k,y} \subseteq \mathscr{V}_{k,y}$, *where the transition operator* $T_{a,M}$ *is defined to be*

$$T_{a,M}v(\alpha) = |\det M| \sum_{\beta \in \mathbb{Z}^s} a(M\alpha - \beta)v(\beta), \quad v \in (\ell_0(\mathbb{Z}^s))^{r\times 1}. \tag{2.18}$$

In fact, if a satisfies the sum rules of order $k+1$ *in* (2.8) *and* (2.9) *with the sequence y, then* $S_{a,M}(p * y) = p(M^{-1}\cdot) * y$ *for all* $p \in \Pi_k$ *and consequently*, $S_{a,M}p - p(M^{-1}\cdot) \in \mathscr{P}_{\deg(p)-1,y}$ *for all* $p \in \mathscr{P}_{k,y}$ *and* $\operatorname{spec}(S_{a,M}|_{\mathscr{P}_{k,y}}) = \operatorname{spec}(T_{a,M}|_{\mathscr{V}_{k,y}}) = \{(\sigma_1, \dots, \sigma_s)^{-\mu}\colon |\mu| \leqslant k, \mu \in \mathbb{N}_0^s\}$, *where* $\operatorname{spec}(M) = \{\sigma_1, \dots, \sigma_s\}$.

Proof. By the definition of $\mathscr{V}_{k,y}$ and $\mathscr{P}_{k,y}$, (1) and (2) hold. Point (3) follows directly from (2.14) and $(p * y) * v = p * (y * v)$. Point (4) can be easily verified by considering the special case $\hat{y}(\xi) = [\hat{y}_1(\xi), 0, \dots, 0]$.

Take $\hat{\tilde{y}}(\xi) = \hat{y}(\xi)\hat{U}_y(\xi)$. By Proposition 2.4, we have $\mathscr{V}_{k,\tilde{y}} = \mathscr{V}_{k,[\tilde{y}_1,0,\dots,0]} = \mathscr{V}_{k,\delta} \times (\ell_0(\mathbb{Z}^s))^{(r-1)\times 1}$. It is known (see [26]) that $\mathscr{V}_{k,\delta} = \operatorname{span}\{\nabla^\mu\delta(\cdot - \beta)\colon \beta \in \mathbb{Z}^s, |\mu| = k+1\}$ which can be proved using long division (see [14,15]). Consequently, we deduce that $\{\nabla^\mu \delta e_1\colon |\mu| = k+1\} \cup \{\delta e_j\colon j = 2, \dots, r\}$ generates $\mathscr{V}_{k,\tilde{y}}$. Now it is easy to see that $\mathscr{B}_{k,y}$ generates $\mathscr{V}_{k,y}$.

To prove (6), by Proposition 2.4, it suffices to prove it for the special case that $\hat{y}(\xi) = [\hat{y}_1(\xi), 0, \dots, 0]$ and $\hat{a}(\xi)$ takes the form of (2.10). Let $b \in \ell_0(\mathbb{Z}^s)$. It is an easy exercise to show that (see [17, Proposition 2.2])

$$\sum_{\beta \in \mathbb{Z}^s} p(\beta)b(\cdot - M\beta) \in \Pi_k \quad \forall p \in \Pi_k \Leftrightarrow D^\mu \hat{b}(2\pi\beta) = 0 \quad \forall |\mu| \leqslant k,\ \beta \in (M^T)^{-1}\mathbb{Z}^s \backslash \mathbb{Z}^s. \tag{2.19}$$

If (2.19) holds and $D^\mu[\hat{w}(M^T\cdot)\hat{b}(\cdot)](0) = D^\mu \hat{w}(0)$ for all $|\mu| \leqslant k$ for some $w \in \ell_0(\mathbb{Z}^s)$, then one has

$$
\begin{aligned}
S_{b,M}p &= |\det M| \sum_{\beta \in \mathbb{Z}^s} p(\beta) b(\cdot - M\beta) = p(M^{-1}\cdot) * b \\
&= \sum_{\mu \in \mathbb{N}_0^s} D^\mu p(M^{-1}\cdot) \frac{(-iM^{-1}D^T)^\mu}{\mu!} \hat{b}(0) \quad \forall p \in \Pi_k
\end{aligned} \tag{2.20}
$$

and consequently $S_{b,M}(p * w) = p(M^{-1}\cdot) * w$ for all $p \in \Pi_k$. In particular, one has

$$
\begin{aligned}
S_{b,M}p = |\det M| \sum_{\beta \in \mathbb{Z}^s} p(\beta) b(\cdot - M\beta) = 0 \quad \forall p \in \Pi_k \\
\Leftrightarrow D^\mu \hat{b}(2\pi\beta) = 0 \quad \forall |\mu| \leqslant k, \beta \in (M^T)^{-1}\mathbb{Z}^s.
\end{aligned} \tag{2.21}
$$

Now by Proposition 2.4, we see that (6) is true since $\mathscr{P}_{k,y} = \{[p, 0, \ldots, 0] \colon p \in \Pi_k\}$ and for all $p \in \Pi_k$,

$$
\begin{aligned}
S_{a,M}[p, 0, \ldots, 0] &= [S_{a_{1,1},M}p, S_{a_{1,2},M}p] = [S_{a_{1,1},M}p, 0, \ldots, 0] \\
&= [p(M^{-1}\cdot) * a_{1,1}, 0, \ldots, 0].
\end{aligned}
$$

Note that when $\hat{\tilde{y}}(\xi) = \hat{y}(\xi)\hat{U}(\xi)$ and $\hat{\tilde{a}}(\xi) = \hat{U}(M^T\xi)^{-1}\hat{a}(\xi)\hat{U}(\xi)$, it is easy to verify that $\widehat{S_{a,M}v}(\xi) = \hat{v}(M^T\xi)\hat{a}(\xi)$ for $u \in (\ell_0(\mathbb{Z}^s))^{1\times r}$ and $S_{\tilde{a},M}(v * \tilde{y}) = [S_{a,M}(v * y)] * U$. We conclude that $S_{a,M}(p * y) = p(M^{-1}\cdot) * y$ for all $p \in \Pi_k$. By (2.13), $S_{a,M}p - p(M^{-1}\cdot) \in \Pi_{\deg(p)-1}$ for all $p \in \Pi_k$ and therefore, $\operatorname{spec}(S_{a,M}|_{\mathscr{P}_{k,y}}) = \{(\sigma_1, \ldots, \sigma_s)^{-\mu} \colon |\mu| \leqslant k\}$.

By a simple computation, for $p \in (\Pi_k)^{1\times r}$ and $v \in (\ell_0(\mathbb{Z}^s))^{r\times 1}$, we have

$$
\begin{aligned}
\sum_{\alpha \in \mathbb{Z}^s} p(\alpha) T_{a,M} v(-\alpha) &= |\det M| \sum_{\alpha, \beta \in \mathbb{Z}^s} p(\alpha) a(-M\alpha - \beta) v(\beta) \\
&= \sum_{\beta \in \mathbb{Z}^s} S_{a,M}p(\beta) v(-\beta).
\end{aligned}
$$

Now (7) follows directly from (6) and the above identity. $\square$

Note that $a_n = |\det M|^{-n} S^n_{a,M}(\delta I_r)$ and

$$
Q^n_{a,M}f = |\det M|^n [a_n * f](M^n\cdot) = [S^n_{a,M}(\delta I_r) * f](M^n\cdot).
$$

3. Initial function vectors in a vector cascade algorithm

In this section, we shall study the initial function vectors in a cascade algorithm. Results in this section will be useful in investigating vector cascade algorithms and refinable function vectors.

In the following, following the lines developed in [18], we study some necessary conditions for initial function vectors in a cascade algorithm. Using Taylor series, we see that the condition in (2.8) is equivalent to saying that $\hat{y}(M^T\xi)\hat{a}(\xi) = \hat{y}(\xi) + o(|\xi|^k)$, as $\xi \to 0$. All the results and proofs involving y in this paper depend only on the values $D^\mu \hat{y}(0), |\mu| \leqslant k$. So, when $D^\mu \hat{\tilde{y}}(0) = D^\mu \hat{y}(0)$ for all $|\mu| \leqslant k$, we can replace y by $\tilde{y}$.

The assumption in (2.8) is justified by the following result which generalizes [4, Lemma 2.1].

Proposition 3.1. *Let M be an $s \times s$ isotropic dilation matrix. Let f be an $r \times 1$ column vector of compactly supported functions in $W_p^k(\mathbb{R}^s)$ such that* $\operatorname{span}\{\hat{f}(2\pi\beta)\colon \beta \in \mathbb{Z}^s\} = \mathbb{C}^{r\times 1}$. *If $\lim_{n\to\infty} \|Q_{a,M}^n f - f_\infty\|_{(W_p^k(\mathbb{R}^s))^{r\times 1}} = 0$ for some $f_\infty \neq 0$, where the cascade operator $Q_{a,M}$ is defined in* (1.2), *then*

$$1 \text{ is a simple eigenvalue of } \hat{a}(0) \text{ and all other eigenvalues of } \hat{a}(0) \text{ are less that } \rho(M)^{-k} \text{ in modulus.} \tag{3.1}$$

Consequently, there exists $y \in (\ell_0(\mathbb{Z}^s))^{1\times r}$ such that $\hat{y}(0) \neq 0$ and (2.8) *holds. Moreover, if* (3.1) *holds, then up to a scalar multiplication the values $D^\mu \hat{y}(0)$, $|\mu| \leqslant k$ satisfying* (2.8) *are uniquely determined by the mask a.*

Proof. Let $f_n := Q_{a,M}^n f$. Then $\hat{f}_n((M^T)^n\xi) = \hat{a}_n(\xi)\hat{f}(\xi)$ and $\lim_{n\to\infty} \|f_n - f_\infty\|_{(W_p^k(\mathbb{R}^s))^{r\times 1}} = 0$, where a_n is defined in (2.7). Note that $\hat{a}_n(0) = [\hat{a}(0)]^n$. It follows from Lemma 2.3 that

$$\begin{aligned} \lim_{n\to\infty} \rho(M)^{kn}[\hat{a}(0)]^n \hat{f}(2\pi\beta) &= \lim_{n\to\infty} \rho(M)^{kn} \hat{f}_n((M^T)^n 2\pi\beta) \\ &= 0 \quad \forall \beta \in \mathbb{Z}^s \backslash \{0\}. \end{aligned} \tag{3.2}$$

We claim that $\hat{f}(0) \neq 0$. Otherwise, combining $\hat{f}(0) = 0$, (3.2) and the assumption $\operatorname{span}\{\hat{f}(2\pi\beta)\colon \beta \in \mathbb{Z}^s\} = \mathbb{C}^{r\times 1}$, we deduce that $\rho(\hat{a}(0)) < \rho(M)^{-k} \leqslant 1$. It follows that $\hat{f}_\infty(\xi) = \lim_{n\to\infty} \hat{a}_n((M^T)^{-n}\xi)\ \hat{f}_\infty((M^T)^{-n}\xi) = 0$ which is a contradiction to our assumption $f_\infty \neq 0$. Now it is easy to verify that (3.1) holds.

Note that (3.1) implies that $[\otimes^j M] \otimes \hat{a}(0)^T - I_{s^j + r}$ is invertible for every $j = 1, \ldots, k$. By Lemma 2.2 and the fact that 1 is a simple eigenvalue of $\hat{a}(0)$, there is a unique solution $\{D^\mu \hat{y}(0)\colon 0 < |\mu| \leqslant k\}$ to the system of linear equations in (2.8) for any given $\hat{y}(0) \neq 0$ satisfying $\hat{y}(0)\hat{a}(0) = \hat{y}(0)$. $\square$

For initial function vectors in a vector cascade algorithm, we have the following result.

Proposition 3.2. *Assume that there exists $y \in (\ell_0(\mathbb{Z}^s))^{1\times r}$ such that $\hat{y}(0) \neq 0$ and* (2.8) *holds. For any compactly supported function vector $f \in (W_p^k(\mathbb{R}^s))^{r\times 1}$, if the sequence $Q_{a,M}^n f$ $(n \in \mathbb{N})$ converges in the Sobolev space $(W_p^k(\mathbb{R}^s))^{r\times 1}$ and*

$\lim_{n\to\infty} \hat{y}(0)\widehat{Q^n_{a,M}f}(0) = 1$, *then*

$$\hat{y}(0)\hat{f}(0) = 1 \quad \textit{and} \quad D^\mu[\hat{y}(\cdot)\hat{f}(\cdot)](2\pi\beta) = 0 \quad \forall |\mu| \leqslant k, \beta \in \mathbb{Z}^s \backslash \{0\}. \tag{3.3}$$

If (3.1) *holds, then there is a unique distribution vector* ϕ *satisfying* $\hat{\phi}(M^T\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$ *and* $\hat{y}(0)\hat{\phi}(0) = 1$. *If* $\phi \in (W_p^k(\mathbb{R}^s))^{r\times 1}$ *satisfies* $\hat{\phi}(M^T\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$ *and* $\hat{y}(0)\hat{\phi}(0) = 1$, *then*

$$D^\mu[\hat{y}(\cdot)\hat{\phi}(\cdot)](0) = \delta(\mu) \quad \textit{and} \quad D^\mu[\hat{y}(\cdot)\hat{\phi}(\cdot)](2\pi\beta) = 0 \quad \forall |\mu| \leqslant k, \beta \in \mathbb{Z}^s \backslash \{0\}. \tag{3.4}$$

Proof. Let $N = M^T$. Define f_n by

$$\hat{f}_n(\xi) := \hat{y}(\xi)\widehat{Q^n_{a,M}f}(\xi) = \hat{y}(\xi)\hat{a}_n(N^{-n}\xi)\hat{f}(N^{-n}\xi).$$

By (2.8) and the Leibniz differentiation formula, for $\beta \in \mathbb{Z}^s$ and $|\mu| \leqslant k$, by induction we have

$$\begin{aligned} D^\mu[\hat{f}_n(N^n\cdot)](2\pi\beta) &= D^\mu[\hat{y}(N^n\cdot)\hat{a}_n(\cdot)\hat{f}(\cdot)](2\pi\beta) \\ &= D^\mu[\hat{y}(N\cdot)\hat{a}(\cdot)\hat{f}(\cdot)](2\pi\beta) = D^\mu[\hat{y}(\cdot)\hat{f}(\cdot)](2\pi\beta). \end{aligned}$$

Since the sequence $Q^n_{a,M}f$ converges in $(W_p^k(\mathbb{R}^s))^{r\times 1}$, we deduce that the sequence f_n converges in $W_p^k(\mathbb{R}^s)$. By Lemma 2.3, we conclude that

$$D^\mu[\hat{y}(\cdot)\hat{f}(\cdot)](2\pi\beta) = \lim_{n\to\infty} D^\mu[\hat{f}_n(N^n\cdot)](2\pi\beta) = 0$$

for all $|\mu| \leqslant k$ and $\beta \in \mathbb{Z}^s \backslash \{0\}$. So, (3.3) holds since $1 = \hat{y}(0)\widehat{Q^n_{a,M}f}(0) = \hat{y}(0)[\hat{a}(0)]^n \hat{f}(0) = \hat{y}(0)\hat{f}(0)$ by $\hat{y}(0)\hat{a}(0) = \hat{y}(0)$.

When (3.1) holds, since 1 is a simple eigenvalue of $\hat{a}(0)$, there is a unique distribution vector ϕ such that $\hat{\phi}(M^T\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$ and $\hat{y}(0)\hat{\phi}(0) = 1$. When $\phi \in (W_p^k(\mathbb{R}^s))^{r\times 1}$, by $Q^n_{a,M}\phi = \phi$, we have $D^\mu[\hat{y}(\cdot)\hat{\phi}(\cdot)](2\pi\beta) = 0$ for all $|\mu| \leqslant k$ and $\beta \in \mathbb{Z}^s \backslash \{0\}$. Since $\hat{\phi}(M^T\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$, by (2.8), we have

$$D^\mu[\hat{y}(M^T\cdot)\hat{\phi}(M^T\cdot)](0) = D^\mu[\hat{y}(M^T\cdot)\hat{a}(\cdot)\hat{\phi}(\cdot)](0) = D^\mu[\hat{y}(\cdot)\hat{\phi}(\cdot)](0) \quad \forall |\mu| \leqslant k.$$

Since M is a dilation matrix, by Lemma 2.2, the above system has a unique solution for $\{D^\mu[\hat{y}(\cdot)\hat{\phi}(\cdot)](0)\colon 0 < |\mu| \leqslant k\}$. Obviously, the above system holds with $D^\mu[\hat{y}(\cdot)\hat{\phi}(\cdot)](0) = \delta(\mu), |\mu| \leqslant k$ which completes the proof. $\square$

For a compactly supported function vector $f \in (W_p^k(\mathbb{R}^s))^{r\times 1}$, we say that f satisfies the *moment conditions* of order $k+1$ with respect to y if (3.3) holds. It is well known that (3.4) is equivalent to $(p * y) * \phi = p$ for all $p \in \Pi_k$. Similarly, (3.3) is equivalent to $p - (p * y) * f \in \Pi_{\deg(p)-1}$ for all $p \in \Pi_k$, where $\deg(p)$ denotes the total degree of p.

Throughout the paper, we denote

$$\begin{aligned}\mathscr{F}_{k,y,p} := \{f \in (W_p^k(\mathbb{R}^s))^{r\times 1}\colon f \text{ is compactly supported and satisfies}\\ \text{the moment conditions of order } k+1\\ \text{with respect to } y \text{ in (3.3)}\}.\end{aligned} \tag{3.5}$$

Note that the set $\mathscr{F}_{k,y,p}$ depends only on the values $D^\mu \hat{y}(0), |\mu| \leqslant k$. One can prove that $Q_{a,M}\mathscr{F}_{k,y,p} \subseteq \mathscr{F}_{k,y,p}$ if and only if a satisfies the sum rules of order $k+1$ in (2.8) and (2.9) with the sequence y.

Lemma 3.3. *Let $y, \tilde{y} \in (\ell_0(\mathbb{Z}^s))^{1\times r}$ such that $\hat{y}(0) \neq 0$ and $\hat{\tilde{y}}(0) \neq 0$. Then $\mathscr{F}_{k,y,p} = \mathscr{F}_{k,\tilde{y},p}$ if and only if there exists $c \in \ell_0(\mathbb{Z}^s)$ such that $\hat{c}(0) = 1$ and*

$$D^\mu \hat{\tilde{y}}(0) = D^\mu[\hat{c}(\cdot)\hat{y}(\cdot)](0) \quad \forall |\mu| \leqslant k. \tag{3.6}$$

Similarly, $\mathscr{V}_{k,y} = \mathscr{V}_{k,\tilde{y}}$ (or $\mathscr{P}_{k,y} = \mathscr{P}_{k,\tilde{y}}$) if and only if there exists $c \in \ell_0(\mathbb{Z}^s)$ such that (3.6) *holds and $\hat{c}(0) \neq 0$.*

Proof. By Proposition 2.4, it suffices to prove it for $\hat{y}(\xi) = [\hat{y}_1(\xi), 0, \dots, 0]$ with $\hat{y}_1(0) = 1$. In this case, $\mathscr{F}_{k,y,p}$ consists of all compactly supported function vectors $[f_1, f_2, \dots, f_r]^T \in (W_p^k(\mathbb{R}^s))^{r\times 1}$ such that $\hat{f_1}(0) = 1$ and $D^\mu \hat{f_1}(2\pi\beta) = 0$ for all $|\mu| \leqslant k$ and $\beta \in \mathbb{Z}^s \backslash \{0\}$. Write $[\hat{\tilde{y}}_1, \dots, \hat{\tilde{y}}_r] = \hat{\tilde{y}}$. Now it is straightforward to see that $\mathscr{F}_{k,y,p} = \mathscr{F}_{k,\tilde{y},p}$ if and only if $\hat{\tilde{y}}_1(0) = 1$ and $D^\mu \hat{\tilde{y}}_j(0) = 0$ for all $|\mu| \leqslant k$ and $j = 2, \dots, r$. It is easy to see that there exists $c \in \ell_0(\mathbb{Z}^s)$ such that $D^\mu \hat{c}(0) = D^\mu[\hat{\tilde{y}}_1(\cdot)/\hat{y}_1(\cdot)](0)$ for all $|\mu| \leqslant k$. We complete the proof. $\square$

The following lemma will be needed later.

Lemma 3.4. *Let $\{c_\mu\colon |\mu| \leqslant k,\ \mu \in \mathbb{N}_0^s\}$ be arbitrarily given complex numbers such that $c_0 = 0$. For any $\varepsilon > 0$, there exists $c \in \ell_0(\mathbb{Z}^s)$ such that $\|\hat{c}(\cdot)\|_{L_\infty} < \varepsilon$ and $D^\mu \hat{c}(0) = c_\mu$ for all $|\mu| \leqslant k$.*

Proof. We prove the claim by induction. When $k = 0$, the claim holds by setting $c = 0$. Suppose that the claim holds for $k = j-1$ with $j \geqslant 1$. By induction hypothesis, there exists $a \in \ell_0(\mathbb{Z}^s)$ such that $\|\hat{a}(\cdot)\|_{L_\infty} < \varepsilon/2$ and $D^\mu \hat{a}(0) = c_\mu$ for all $|\mu| \leqslant j-1$. It is easy to see that there exists $b \in \ell_0(\mathbb{Z}^s)$ such that $D^\mu \hat{b}(0) = 0$ for all $|\mu| \leqslant j-1$ and $D^\mu \hat{b}(0) = c_\mu - D^\mu \hat{a}(0)$ for all $|\mu| = j$. For a large enough integer n, we see that $\|n^{-j}\hat{b}(n\cdot)\|_{L_\infty} < \varepsilon/2$. Set $\hat{c}(\xi) = \hat{a}(\xi) + n^{-j}\hat{b}(n\xi)$. Then $c \in \ell_0(\mathbb{Z}^s)$ is desired since it is easy to verify that $\|\hat{c}(\cdot)\|_{L_\infty} < \varepsilon$ and $D^\mu \hat{c}(0) = c_\mu$ for all $|\mu| \leqslant j$. So, the claim holds for $k = j$. The proof is completed by induction. $\square$

For an $r \times 1$ vector f of compactly supported distributions on $\mathbb{R}^s$, we say that the shifts of f are *linearly independent* if $\operatorname{span}\{\hat{f}(\xi + 2\pi\beta)\colon \beta \in \mathbb{Z}^s\} = \mathbb{C}^{r\times 1}$ for all $\xi \in \mathbb{C}^{r\times 1}$.

Therefore, if the shifts of f are linearly independent, then the shifts of f are stable. Let f be a compactly supported function vector in $(L_p(\mathbb{R}^s))^{r\times 1}$. It is known (see [30]) that the shifts of f are stable if and only if there exist positive constants C_1 and C_2 such that

$$C_1\|v\|_{(\ell_p(\mathbb{Z}^s))^{1\times r}} \leqslant \left\| \sum_{\beta\in\mathbb{Z}^s} v(\beta)f(\cdot-\beta)\right\|_{L_p(\mathbb{R}^s)} \leqslant C_2\|v\|_{(\ell_p(\mathbb{Z}^s))^{1\times r}} \quad \forall v\in(\ell_p(\mathbb{Z}^s))^{1\times r}. \tag{3.7}$$

Note that when $f\in(L_p(\mathbb{R}^s))^{r\times 1}$ and f is compactly supported, it can be easily proved that the right side of (3.7) holds for some positive constant C_2. For $v=(v_1,\ldots,v_s)$ and $\mu=(\mu_1,\ldots,\mu_s)$, $v\leqslant\mu$ means $v_j\leqslant\mu_j$ for all $j=1,\ldots,s$, and $v<\mu$ means $v\leqslant\mu$ and $v\neq\mu$.

Before proceeding further, let us answer the question in [4] (see Q1 in Section 1 for more detail) by the following stronger result.

Proposition 3.5. *Let $y\in(\ell_0(\mathbb{Z}^s))^{1\times r}$ such that $\hat{y}(0)\neq 0$. Let b_μ $(0<|\mu|\leqslant J)$ be any complex numbers. Then there is an $r\times 1$ compactly supported function vector f in $(C^J(\mathbb{R}^s))^{r\times 1}$ such that the shifts of f are stable, $D^\mu[\hat{y}(\cdot)\hat{f}(\cdot)](0)=b_\mu$ for all $0<|\mu|\leqslant J$, and f satisfies the moment conditions of order $J+1$ in* (3.3) *with respect to y. Moreover, without the requirement that $D^\mu[\hat{y}(\cdot)\hat{f}(\cdot)](0)=b_\mu$ for all $0<|\mu|\leqslant J$, the shifts of f can be linearly independent.*

Proof. By Proposition 2.4, it suffices to prove the claim for $\hat{y}(\xi)=[\hat{y}_1(\xi),0,\ldots,0]$ with $\hat{y}_1(0)=1$. It is well known [9] that there is a univariate compactly supported orthogonal $(r+1)$-refinable function $\phi\in C^J(\mathbb{R})$ and there exist compactly supported C^J wavelet functions $\psi_1,\ldots,\psi_r$ such that $\{\phi(\cdot-\beta),\psi_j(\cdot-\beta)\colon j=1,\ldots,r;\beta\in\mathbb{Z}\}$ is an orthogonal system. By Proposition 3.2, $\hat{\phi}(0)=1$ and $D^\mu\hat{\phi}(2\pi\beta)=0$ for all $0\leqslant\mu\leqslant J$ and $\beta\in\mathbb{Z}\backslash\{0\}$. Now we take the tensor product in $\mathbb{R}^s$. So, we have an $(r+1)I_s$-refinable function Φ and wavelet functions $\Psi_1,\ldots,\Psi_{(r+1)^s-1}$ such that their shifts are orthogonal. It is clear that $\hat{\Phi}(0)=1$ and $D^\mu\hat{\Phi}(2\pi\beta)=0$ for all $|\mu|\leqslant J$ and $\beta\in\mathbb{Z}^s\backslash\{0\}$. Let $c_0=1$ and recursively define

$$c_\mu := b_\mu - \sum_{0\leqslant v<\mu}\frac{\mu!}{v!(\mu-v)!}D^{\mu-v}[\hat{y}_1(\cdot)\hat{\Phi}(\cdot)](0)c_v, \quad 0<|\mu|\leqslant J.$$

By Lemma 3.4, there exists $c\in\ell_0(\mathbb{Z}^s)$ such that $\hat{c}(0)=0$, $D^\mu\hat{c}(0)=c_\mu$ for all $0<|\mu|\leqslant J$ and $\|\hat{c}(\cdot)\|_{L_\infty}\leqslant 1/2$. Now define f_1 by $\hat{f}_1(\xi)=(1+\hat{c}(\xi))\hat{\Phi}(\xi)$ and $f_j=\Psi_{j-1}$ for all $j=2,\ldots,r$. By the Leibniz differentiation formula, it is easy to see that f is desired since $1+\hat{c}(\xi)\neq 0$ for all $\xi\in\mathbb{R}^s$ and $\{\Phi(\cdot-\beta)\colon\beta\in\mathbb{Z}^s\}\cup\{f_j(\cdot-\beta)\colon j=2,\ldots,r;\beta\in\mathbb{Z}^s\}$ is an orthogonal system and therefore stable. $\square$

We observe that the function vector f in Proposition 3.5 can also be constructed similarly from other suitable scalar refinable functions such as the B-spline functions using biorthogonal bases rather than orthogonal bases.

For $\alpha \in \mathbb{Z}^s$ and $t \in \mathbb{R}^s$, we define

$$\begin{aligned} &\nabla_\alpha v := v - v(\cdot - \alpha), \\ &\nabla_t f := f - f(\cdot - t), \quad v \in (\ell_0(\mathbb{Z}^s))^{m\times n}, \quad f \in (L_p(\mathbb{R}^s))^{m\times n}. \end{aligned} \tag{3.8}$$

For $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$, $\nabla^\mu := \nabla_{e_1}^{\mu_1} \cdots \nabla_{e_s}^{\mu_s}$, where e_j is the jth coordinate unit vector in $\mathbb{R}^s$. Note that $\nabla^\mu v = \nabla^\mu \delta * v$ and $\nabla_\alpha f = \nabla_\alpha \delta * f$ for $\alpha \in \mathbb{Z}^s$.

Following the lines developed in [18], in the rest of this section we investigate the mutual relations among the initial function vectors in a vector cascade algorithm.

Theorem 3.6. *Let $y \in (\ell_0(\mathbb{Z}^s))^{1\times r}$ such that $\hat{y}(0) \neq 0$. Let f be a compactly supported function vector in $(L_p(\mathbb{R}^s))^{r\times 1}$, where $0 < p \leqslant \infty$. Then for any nonnegative integer k, the following statements are equivalent*:

(1) *$D^\mu[\hat{y}(\cdot)\hat{f}(\cdot)](2\pi\beta) = 0$ for all $\beta \in \mathbb{Z}^s$ and $\mu \in \mathbb{N}_0^s$ with $|\mu| \leqslant k$;*
(2) *$\sum_{\beta\in\mathbb{Z}^s} p(\beta) f(\cdot - \beta) = p * f = 0$ for all $p \in \mathscr{P}_{k,y}$, where $\mathscr{P}_{k,y}$ is defined in* (2.15);
(3) *For some positive integer N_f, $f = \sum_{j=1}^{N_f} v_j * g_j$ for some compactly supported functions $g_j \in L_p(\mathbb{R}^s)$ and some $v_j \in \mathscr{V}_{k,y}$, where $\mathscr{V}_{k,y}$ is defined in* (2.14);
(4) *$f = \sum_{v\in\mathscr{B}_{k,y}} v * g_v$ for some compactly supported functions $g_v \in L_p(\mathbb{R}^s)$, where $\mathscr{B}_{k,y}$ is defined in* (2.16).

Proof. By Proposition 2.4, it suffices to prove the claim for the case $\hat{y}(\xi) = [\hat{y}_1(\xi), 0, \ldots, 0]$. For this special y, we observe that $\mathscr{P}_{k,y} = \{[p, 0, \ldots, 0]: p \in \Pi_k\}$ and $\mathscr{V}_{k,y} = \mathscr{V}_{k,\delta} \times (\ell_0(\mathbb{Z}^s))^{(r-1)\times 1}$.

Let g be the first component in the vector f. Since g is compactly supported, the linear space $\operatorname{span}\{g(\cdot - \beta)\chi_{[0,1]^s}: \beta \in \mathbb{Z}^s\}$ is finite dimensional. So pick up a basis $g_1, \ldots, g_N$ for this space from the set $\{g(\cdot - \beta)\chi_{[0,1]^s}: \beta \in \mathbb{Z}^s\}$. Then the function g can be uniquely written as the following finite sum $g = \sum_{j=1}^N v_j * g_j$ for some $v_j \in \ell_0(\mathbb{Z}^s)$. By a simple computation, it is easy to verify (see [18, Theorem 1] for more detail) that $\sum_{\beta\in\mathbb{Z}^s} p(\beta) g(\cdot - \beta) = 0$ for all $p \in \Pi_k$ if and only if $v_j \in \mathscr{V}_{k,\delta}$ for all $j = 1, \ldots, N$. Note that for this particular form of y, (1) is equivalent to $D^\mu \hat{g}(2\pi\beta) = 0$ for all $|\mu| \leqslant k$ and $\beta \in \mathbb{Z}^s$. This completes the proof. $\square$

There is a similar result of Theorem 3.6 on sequences [15]. For $b \in \ell_0(\mathbb{Z}^s)$, $D^\mu \hat{b}(2\pi\beta) = 0$ for all $|\mu| \leqslant k$ and $\beta \in (M^T)^{-1}\mathbb{Z}^s$ if and only if $\hat{b}(\xi) = \sum_{|\mu|=k+1} \widehat{\nabla^\mu \delta}(M^T\xi)\hat{u}_\mu(\xi)$ for some $u_\mu \in \ell_0(\mathbb{Z}^s)$, or equivalently, $\hat{b}(\xi) = \sum_{j=1}^{N_a} \hat{v}_j(M^T\xi)\hat{u}_j(\xi)$ for some $N_a \in \mathbb{N}$ and some sequences $v_j \in \mathscr{V}_{k,\delta}$ and $u_j \in \ell_0(\mathbb{Z}^s)$ for $j = 1, \ldots, N_a$.

As a direct consequence of Theorem 3.6, we have (see [18, Corollary 2.2]) the following result.

Corollary 3.7. *Let* $y\in(\ell_0(\mathbb{Z}^s))^{1\times r}$ *such that* $\hat{y}(0)\neq 0$. *If* $f,g\in\mathscr{F}_{k,y,p}$, *where* $\mathscr{F}_{k,y,p}$ *is defined in* (3.5), *then*

$$D^\mu g = D^\mu f + \sum_{v\in\mathscr{B}_{|\mu|,y}} v * h_{\mu,v} \quad \text{and}$$

$$D^\mu f = \sum_{v\in\mathscr{B}_{|\mu|-1,y}} v * H_{\mu,v} \quad \forall |\mu|\leqslant k,\ \mu\in\mathbb{N}_0^s \tag{3.9}$$

for some compactly supported functions $h_{\mu,v}, H_{\mu,v}\in L_p(\mathbb{R}^s)$, *or equivalently,*

$$[\otimes^j D]\otimes g = [\otimes^j D]\otimes f + \sum_{v\in\mathscr{B}_{j,y}} v * h_{j,v} \quad \text{and}$$

$$[\otimes^j D]\otimes f = \sum_{v\in\mathscr{B}_{j-1,y}} v * H_{j,v}, \quad j=0,\ldots,k$$

for some compactly supported function vectors $h_{j,v}, H_{j,v}\in(L_p(\mathbb{R}^s))^{1\times s^j}$.

4. Convergence of vector cascade algorithms in Sobolev spaces

In this section, we shall characterize convergence of a vector cascade algorithm in a Sobolev space and we shall settle the question Q2 in Section 1. Before proceeding further, let us introduce a very important quantity. Let a be a matrix mask with multiplicity r. For any $y\in(\ell_0(\mathbb{Z}^s))^{1\times r}$ such that $\hat{y}(0)\neq 0$, we define

$$\rho_k(a,M,p,y) := \sup\left\{ \lim_{n\to\infty} \|a_n * v\|^{1/n}_{(\ell_p(\mathbb{Z}^s))^{r\times 1}}:\ v\in\mathscr{V}_{k,y}\right\}, \quad 1\leqslant p\leqslant\infty, \tag{4.1}$$

where a_n is defined in (2.7) and $\mathscr{V}_{k,y}$ is defined in (2.14). Let $\mathscr{B}_{k,y}$ be defined in (2.16). By Proposition 2.4, we see that

$$\rho_k(a,M,p,y) = \max\left\{ \lim_{n\to\infty} \|a_n * v\|^{1/n}_{(\ell_p(\mathbb{Z}^s))^{r\times 1}}:\ v\in\mathscr{B}_{k,y}\right\}$$

since $\mathscr{B}_{k,y}$ generates $\mathscr{V}_{k,y}$ and $\|a_n*(v(\cdot-\beta))\|_{(\ell_p(\mathbb{Z}^s))^{r\times 1}} = \|a_n * v\|_{(\ell_p(\mathbb{Z}^s))^{r\times 1}}$ for all $\beta\in\mathbb{Z}^s$. Define

$$\rho(a,M,p) := \inf\{\rho_k(a,M,p,y):\ (2.8)\text{ and }(2.9)\text{ hold for some } k\in\mathbb{N}_0 \text{ and some } y\in(\ell_0(\mathbb{Z}^s))^{1\times r} \text{ with } \hat{y}(0)\neq 0\}. \tag{4.2}$$

We define the following important quantity:

$$\nu_p(a,M) := -\log_{\rho(M)}[|\det M|^{1-1/p}\rho(a,M,p)], \quad 1\leqslant p\leqslant\infty. \tag{4.3}$$

The above quantity $\nu_p(a,M)$ plays a very important role in characterizing the convergence of a vector cascade algorithm in a Sobolev space and in characterizing the L_p smoothness of a refinable function vector.

The quantity $\rho_k(a, M, p, y)$ defined in (4.1) can be rewritten using the ℓ_p-norm joint spectral radius. Let $\mathscr{A}$ be a finite collection of linear operators acting on a finite-dimensional normed vector space V. For a positive integer n, $\mathscr{A}^n$ denotes $\mathscr{A}^n = \{(A_1, \ldots, A_n)\colon A_1, \ldots, A_n \in \mathscr{A}\}$, and for $1 \leqslant p \leqslant \infty$, we define

$$\|\mathscr{A}^n\|_p^p := \sum_{(A_1,\ldots,A_n)\in\mathscr{A}^n} \|A_1 \cdots A_n\|^p$$

and

$$\|\mathscr{A}^n\|_\infty := \max\{\|A_1 \cdots A_n\| : (A_1, \ldots, A_n) \in \mathscr{A}^n\},$$

where $\|\cdot\|$ denotes any operator norm. For $1 \leqslant p \leqslant \infty$, the *$\ell_p$-norm joint spectral radius* of $\mathscr{A}$ (see [4,11,15,21,25,31,46] and references therein) is defined to be

$$\rho_p(\mathscr{A}) := \lim_{n\to\infty} \|\mathscr{A}^n\|_p^{1/n} = \inf_{n\geqslant 1} \|\mathscr{A}^n\|_p^{1/n}. \tag{4.4}$$

Let Γ_M be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M\mathbb{Z}^s$. To relate the quantity $\rho_k(a, M, p, y)$ to the ℓ_p-norm joint spectral radius, we introduce $A_\varepsilon (\varepsilon \in \Gamma_M)$ on $(\ell_0(\mathbb{Z}^s))^{r\times 1}$ by

$$A_\varepsilon v(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(M\alpha - \beta + \varepsilon)v(\beta), \quad v \in (\ell_0(\mathbb{Z}^s))^{r\times 1},\ \alpha \in \mathbb{Z}^s. \tag{4.5}$$

It was proved in [21, Lemma 2.3] that if a is finitely supported, then for any finitely supported sequence v on $\mathbb{Z}^s$, there exists a finite-dimensional subspace $V(v)$ of $(\ell_0(\mathbb{Z}^s))^{r\times 1}$ such that $V(v)$ contains v and $V(v)$ is the smallest subspace of $(\ell_0(\mathbb{Z}^s))^{r\times 1}$ which is invariant under the operators A_ε, $\varepsilon \in \Gamma_M$. We call such $V(v)$ the minimal $\{A_\varepsilon\colon \varepsilon \in \Gamma_M\}$ invariant subspace generated by v.

Let $\mathscr{A} := \{A_\varepsilon|_W\colon \varepsilon \in \Gamma_M\}$ where W is the minimal $\{A_\varepsilon\colon \varepsilon \in \Gamma_M\}$ invariant subspace generated by a finite subset $\mathscr{B}$ of $(\ell_0(\mathbb{Z}^s))^{r\times 1}$. By [21, Lemmas 2.2 and 2.4], there exists a positive constant C such that

$$C^{-1}\|\mathscr{A}^n\|_p \leqslant \max\{\|a_n * v\|_{(\ell_p(\mathbb{Z}^s))^{r\times 1}}\colon v \in \mathscr{B}\} \leqslant C\|\mathscr{A}^n\|_p \quad \forall n \in \mathbb{N}. \tag{4.6}$$

Consequently, when $\mathscr{B} = \mathscr{B}_{k,y}$, $\rho_k(a, M, p, y) = \rho_p(\mathscr{A})$. Moreover, since

$$|\det M|^{n(1/p-1/q)}\|\mathscr{A}^n\|_q \leqslant \|\mathscr{A}^n\|_p \leqslant \|\mathscr{A}^n\|_q$$

(see [21]) for $1 \leqslant q \leqslant p \leqslant \infty$ and $n \in \mathbb{N}$, it follows that

$$\begin{aligned} |\det M|^{1/q-1/p}\rho_k(a, M, p, y) &\leqslant \rho_k(a, M, q, y) \\ &\leqslant \rho_k(a, M, p, y), \quad 1 \leqslant p \leqslant q \leqslant \infty,\ k \in \mathbb{N}. \end{aligned}$$

In other words, we have

$$\begin{aligned} v_p(a, M) \geqslant v_q(a, M) \geqslant v_p(a, M) + (1/q - 1/p)\log_{\rho(M)}|\det M|,& \\ 1 \leqslant p \leqslant q \leqslant \infty.& \end{aligned} \tag{4.7}$$

Now we have the following result which generalizes [18, Proposition 2.7].

Proposition 4.1. *Let M be an $s\times s$ dilation matrix. Let a be a finitely supported mask on $\mathbb{Z}^s$ with multiplicity r. Let $v_1,\dots,v_J\in(\ell_0(\mathbb{Z}^s))^{r\times 1}$. Then for any $\rho>0$ and $1\leqslant p\leqslant\infty$,*

$$\lim_{n\to\infty}\rho^n\|a_n*v_j\|_{(\ell_p(\mathbb{Z}^s))^{r\times 1}}=0\quad\forall j=1,\dots,J \tag{4.8}$$

if and only if there exist $0<\rho_0<1$ and a positive constant C such that

$$\|a_n*v_j\|_{(\ell_p(\mathbb{Z}^s))^{r\times 1}}\leqslant C\rho^{-n}\rho_0^n\quad\forall n\in\mathbb{N}, j=1,\dots,J. \tag{4.9}$$

Moreover, assume that (2.8) *holds for some $k\in\mathbb{N}_0$ and $y\in(\ell_0(\mathbb{Z}^s))^{1\times r}$ with $\hat{y}(0)\neq 0$. If* $\operatorname{span}\{\hat{v}_j(2\pi\beta)\colon j=1,\dots,J\}=\mathbb{C}^{r\times 1}$ *for all $\beta\in(M^T)^{-1}\mathbb{Z}^s\backslash\mathbb{Z}^s$ and* (4.8) *holds with* $\rho=|\det M|^{1-1/p}\rho(M)^k$, *then the mask a must satisfy the sum rules of order at least $k+1$ in* (2.9) *with the sequence y. In particular, if $\rho_j(a,M,p,\tilde{y})<|\det M|^{1/p-1}\rho(M)^{-k}$ for some $1\leqslant p\leqslant\infty$, $j\in\mathbb{N}_0$ and $\tilde{y}\in(\ell_0(\mathbb{Z}^s))^{1\times r}$ with $\hat{\tilde{y}}(0)\neq 0$, then a must satisfy the sum rules of order $k+1$ in* (2.8) *and* (2.9) *with the sequence y, and one must have $j\geqslant k$ and $\mathscr{V}_{k,\tilde{y}}=\mathscr{V}_{k,y}$.*

Proof. Let us use a similar technique as in the proof of [18, Proposition 2.7]. With the help of ℓ_p-norm joint spectral radius and the relations in (4.4) and (4.6), we see that (4.8) is equivalent to (4.9).

Denote $N:=M^T$. Suppose that a satisfies the sum rules of order L in (2.8) and (2.9) with y for $0\leqslant L<k+1$ (Obviously, it is true when $L=0$). By (2.8), for all $\beta\in N^{-1}\mathbb{Z}^s\backslash\mathbb{Z}^s$ and $|\mu|=L$, by induction we have

$$\begin{aligned}D^\mu[\hat{y}(N^n\cdot)\hat{a}_n(\cdot)\hat{v}_j(\cdot)](2\pi\beta)&=D^\mu[\hat{y}(N\cdot)\hat{a}(\cdot)\hat{v}_j(\cdot)](2\pi\beta)\\&=D^\mu[\hat{y}(N\cdot)\hat{a}(\cdot)](2\pi\beta)\hat{v}_j(2\pi\beta),\end{aligned}$$

where in the last identity we used the induction hypothesis $D^\nu[\hat{y}(N\cdot)\hat{a}(\cdot)](2\pi\beta)=0$ for all $|\nu|<|\mu|=L$ and $\beta\in N^{-1}\mathbb{Z}^s\backslash\mathbb{Z}^s$. Using the same technique as in the proof of [18, Proposition 2.7], one can show that (4.9) implies that for $\beta\in N^{-1}\mathbb{Z}^s\backslash\mathbb{Z}^s$ and $|\mu|=L$,

$$\lim_{n\to\infty}D^\mu[\hat{y}(N\cdot)\hat{a}(\cdot)](2\pi\beta)\hat{v}_j(2\pi\beta)=\lim_{n\to\infty}D^\mu[\hat{y}(N^n\cdot)\hat{a}_n(\cdot)\hat{v}_j(\cdot)](2\pi\beta)=0.$$

Since $\operatorname{span}\{\hat{v}_j(2\pi\beta)\colon j=1,\dots,J\}=\mathbb{C}^{r\times 1}$ for all $\beta\in N^{-1}\mathbb{Z}^s\backslash\mathbb{Z}^s$, we see that $D^\mu[\hat{y}(N\cdot)\hat{a}(\cdot)](2\pi\beta)=0$ for all $|\mu|=L$ and $\beta\in N^{-1}\mathbb{Z}^s\backslash\mathbb{Z}^s$. By induction, a must satisfy the sum rules of order $k+1$ in (2.8) and (2.9) with y.

When $\rho_j(a,M,p,\tilde{y})<|\det M|^{1/p-1}\rho(M)^{-k}$, by Proposition 2.4, we see that $\operatorname{span}\{\hat{v}(2\pi\beta)\colon v\in\mathscr{B}_{j,\tilde{y}}\}=\mathbb{C}^{r\times 1}$ for all $\beta\in N^{-1}\mathbb{Z}^s\backslash\mathbb{Z}^s$. So a must satisfy the sum rules of order $k+1$ in (2.8) and (2.9) with the sequence y. By (2.8), we have $D^\mu[\hat{y}(N^n\cdot)\hat{a}_n(\cdot)\hat{v}(\cdot)](0)=D^\mu[\hat{y}(\cdot)\hat{v}(\cdot)](0)$ for all $|\mu|\leqslant k$. By (4.9), we conclude that

$$D^\mu[\hat{y}(\cdot)\hat{v}(\cdot)](0)=\lim_{n\to\infty}D^\mu[\hat{y}(N^n\cdot)\hat{a}_n(\cdot)\hat{v}(\cdot)](0)=0\quad\forall|\mu|\leqslant k, v\in\mathscr{V}_{j,\tilde{y}}.$$

Therefore, by the definition of $\mathscr{V}_{k,y}$, $\mathscr{V}_{j,\tilde{y}}\subseteq\mathscr{V}_{k,y}$. In the following, we show that $\mathscr{V}_{j,\tilde{y}}\subseteq\mathscr{V}_{k,y}$ implies that $j\geqslant k$ and $\mathscr{V}_{k,\tilde{y}}=\mathscr{V}_{k,y}$. By Proposition 2.4, we can assume that $\hat{y}(\xi)=[\hat{y}_1(\xi),0,\dots,0]$ with $\hat{y}_1(0)=1$. So $\mathscr{V}_{k,y}=\mathscr{V}_{k,\delta}\times(\ell_0(\mathbb{Z}^s))^{(r-1)\times 1}$. It is trivial to see that $\{[v, 0, \dots, 0]^T\colon v \in \mathscr{V}_{j,\delta}\} \subseteq \mathscr{V}_{j,\tilde{y}} \subseteq \mathscr{V}_{k,y}$ and therefore, $\mathscr{V}_{j,\delta} \subseteq \mathscr{V}_{k,\delta}$. Hence, we must have $j \geqslant k$.

Denote $[\hat{\tilde{y}}_1(\xi), \dots, \hat{\tilde{y}}_r(\xi)] = \hat{\tilde{y}}(\xi)$. In the following, we show that

$$\hat{\tilde{y}}_1(0) \neq 0 \quad \text{and} \quad D^\mu \hat{\tilde{y}}_\ell(0) = 0 \quad \forall |\mu| \leqslant k, \ \ \ell = 2, \dots, r. \tag{4.10}$$

Suppose that $\hat{\tilde{y}}_\ell(0) \neq 0$ for some $2 \leqslant \ell \leqslant r$. Say, $\hat{\tilde{y}}_2(0) \neq 0$. There exists $v_2 \in \ell_0(\mathbb{Z}^s)$ such that

$$D^\mu \hat{v}_2(0) = -D^\mu [\hat{\tilde{y}}_1(\cdot)/\hat{\tilde{y}}_2(\cdot)](0) \quad \forall |\mu| \leqslant j,$$

that is,

$$D^\mu [\hat{\tilde{y}}_1(\cdot) + \hat{v}_2(\cdot)\hat{\tilde{y}}_2(\cdot)](0) = 0 \quad \forall |\mu| \leqslant j.$$

So $[\delta, v_2, 0, \dots, 0]^T \in \mathscr{V}_{j,\tilde{y}} \subseteq \mathscr{V}_{k,y}$ which is a contradiction since $\delta \notin \mathscr{V}_{k,\delta}$. Therefore, we conclude that $\hat{\tilde{y}}_\ell(0) = 0$ for all $\ell = 2, \dots, r$. Since $\hat{\tilde{y}}(0) \neq 0$, we must have $\hat{\tilde{y}}_1(0) \neq 0$. Since $\hat{\tilde{y}}_1(0) \neq 0$, there exists $v_1 \in \ell_0(\mathbb{Z}^s)$ such that

$$\begin{aligned} &D^\mu \hat{v}_1(0) = -D^\mu [\hat{\tilde{y}}_2(\cdot)/\hat{\tilde{y}}_1(\cdot)](0), \quad \text{that is,} \\ &D^\mu [\hat{v}_1(\cdot)\hat{\tilde{y}}_1(\cdot) + \hat{\tilde{y}}_2(\cdot)](0) = 0 \quad \forall |\mu| \leqslant j. \end{aligned} \tag{4.11}$$

So, $[v_1, \delta, 0, \dots, 0]^T \in \mathscr{V}_{j,\tilde{y}} \subseteq \mathscr{V}_{k,y}$ which implies $v_1 \in \mathscr{V}_{k,\delta}$; that is, $D^\mu \hat{v}_1(0) = 0$ for all $|\mu| \leqslant k$. Since $j \geqslant k$, it follows from (4.11) that

$$D^\mu \hat{\tilde{y}}_2(0) = -D^\mu [\hat{v}_1(\cdot)\hat{\tilde{y}}_1(\cdot)](0) = 0$$

for all $|\mu| \leqslant k$. Similarly, we can prove that $D^\mu \hat{\tilde{y}}_\ell(0) = 0$ for all $|\mu| \leqslant k$ and $\ell = 2, \dots, r$. So, (4.10) holds. Now it follows directly from (4.10) that $\mathscr{V}_{k,\tilde{y}} = \mathscr{V}_{k,y}$. $\square$

By a similar argument as in [19, Theorem 3.1], if a satisfies the sum rules of order $k+1$ with some $y \in (\ell_0(\mathbb{Z}^s))^{1\times r}$, then

$$\rho_j(a, M, p, y) = \max\{\rho_k(a, M, p, y), \ |\det M|^{1/p-1} \rho(M^{-1})^{j+1}\}$$

for all $1 \leqslant p \leqslant \infty$ and $0 \leqslant j \leqslant k$.

In order to investigate vector cascade algorithms in Sobolev spaces, we need the following result which is essentially known in approximation theory (see Jia [27] and cf. [6]).

Lemma 4.2. *Let M be an $s \times s$ isotropic dilation matrix. Let g be a compactly supported function in $W_p^k(\mathbb{R}^s)$ (when $p = \infty$, replace $W_p^k(\mathbb{R}^s)$ by $C^k(\mathbb{R}^s)$) such that $\hat{g}(0) \neq 0$ and $D^\mu \hat{g}(2\pi\beta) = 0$ for all $|\mu| \leqslant k$ and $\beta \in \mathbb{Z}^s \backslash \{0\}$, then for any compactly* *supported function* $f \in W_p^k(\mathbb{R}^s)$,

$$\inf_{v \in \ell_0(\mathbb{Z}^s)} \left\| f - \sum_{\beta \in \mathbb{Z}^s} v(\beta) g(M^n \cdot - \beta) \right\|_{L_p(\mathbb{R}^s)} \leqslant C\rho(M^{-n})^k \sum_{\mu \in \mathbb{N}_0^s, |\mu|=k} \omega_p(D^\mu f, \rho(M^{-n})) \quad \forall n \in \mathbb{N},$$

where $C > 0$ *is independent of* f *and* n, *and* $\omega_p(f,h) := \sup_{\|t\| \leqslant h} \|f - f(\cdot - t)\|_{L_p(\mathbb{R}^s)}$, $h > 0$.

Now we have the main result in this section which characterizes the convergence of a vector cascade algorithm in a Sobolev space in various ways.

Theorem 4.3. *Let* M *be an* $s \times s$ *isotropic dilation matrix and* Γ_M *be a complete set of representatives of the distinct cosets of* $\mathbb{Z}^s/M\mathbb{Z}^s$. *Let a be a finitely supported matrix mask on* $\mathbb{Z}^s$ *with multiplicity r. Assume that there is a sequence* $y \in (\ell_0(\mathbb{Z}^s))^{1 \times r}$ *such that* $\hat{y}(0) \neq 0$ *and* (2.8) *holds for a nonnegative integer* k. *Then the following statements are equivalent*:

(1) *For every* $f \in \mathscr{F}_{k,y,p}$, *where* $\mathscr{F}_{k,y,p}$ *is defined in* (3.5), *the cascade algorithm with mask* a, *dilation matrix* M *and the initial function vector* f *converges in* $(W_p^k(\mathbb{R}^s))^{r \times 1}$; *that is,* $Q_{a,M}^n f$ $(n \in \mathbb{N})$ *is a Cauchy sequence in the Sobolev space* $(W_p^k(\mathbb{R}^s))^{r \times 1}$;

(2) *For some* $f \in \mathscr{F}_{k,y,p}$ (*When* $p = \infty$, f *is required to be in* $(C^k(\mathbb{R}^s))^{r \times 1}$) *such that the shifts of* f *are stable* (*the existence of such an initial function vector* f *is guaranteed by Proposition* 3.5), *the cascade algorithm with mask* a, *dilation matrix* M *and the initial function vector* f *converges in* $(W_p^k(\mathbb{R}^s))^{r \times 1}$;

(3) $\lim_{n \to \infty} |\det M|^{(1-1/p)n} \rho(M)^{nk} \|a_n * v\|_{(\ell_p(\mathbb{Z}^s))^{r \times 1}} = 0$ *for all* $v \in \mathscr{B}_{k,y}$, *where* a_n *is defined in* (2.7) *and* $\mathscr{B}_{k,y}$ *is defined in* (2.16);

(4) $\lim_{n \to \infty} |\det M|^{(1-1/p)n} \rho(M)^{nk} \|a_n * v\|_{(\ell_p(\mathbb{Z}^s))^{r \times 1}} = 0$ *for all* $v \in \mathscr{V}_{k,y}$, *where* $\mathscr{V}_{k,y}$ *is defined in* (2.14);

(5) $\rho_k(a, M, p, y) < |\det M|^{1/p-1} \rho(M)^{-k}$, *where* $\rho_k(a, M, p, y)$ *is defined in* (4.1);

(6) $\rho_k(a, M, p, y) < |\det M|^{1/p-1} \rho(M)^{-k}$ *and the mask a satisfies the sum rules of order* $k+1$ *in* (2.8) *and* (2.9) *with the sequence* y;

(7) $\rho(a, M, p) < |\det M|^{1/p-1} \rho(M)^{-k}$, *where* $\rho(a, M, p)$ *is defined in* (4.2);

(8) $v_p(a, M) > k$, *where* $v_p(a, M)$ *is defined in* (4.3);

(9) $\rho_p(\{A_\varepsilon|_W\colon \varepsilon \in \Gamma_M\}) < |\det M|^{1/p-1} \rho(M)^{-k}$, *where the operators* A_ε *are defined in* (4.5) *and* W *is the minimal* $\{A_\varepsilon\colon \varepsilon \in \Gamma_M\}$ *invariant subspace generated by* $\{v\colon v \in \mathscr{B}_{k,y}\}$;

(10) $\mathscr{V}_{k,y}$ *is invariant under all the operators* A_ε, $\varepsilon \in \Gamma_M$ *and*

$$\rho_p(\{A_\varepsilon|_{\mathscr{V}_{k,y} \cap (\ell(K))^{r\times 1}}: \varepsilon \in \Gamma_M\}) < |\det M|^{1/p-1} \rho(M)^{-k},$$

where $K := \mathbb{Z}^s \cap \sum_{j=1}^\infty M^{-j} K_0$ *and*

$$K_0 := \{0, \alpha \in \mathbb{Z}^s:\ a(\alpha) \neq 0\} - \Gamma_M + \{\alpha \in \mathbb{Z}^s:\ |\alpha| \leqslant 1\}.$$

Moreover, any of the above statements implies that (3.1) *holds and there is a unique compactly supported function vector* $\phi \in (W_p^k(\mathbb{R}^s))^{r\times 1}$ *such that* $\hat{y}(0)\hat{\phi}(0) = 1$, $\hat{\phi}(M^T\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$ *and* $\lim_{n\to\infty} \|Q_{a,M}^n f - \phi\|_{(W_p^k(\mathbb{R}^s))^{r\times 1}} = 0$ *for every initial function vector* $f \in \mathscr{F}_{k,y,p}$.

Proof. Obviously, (1) $\Rightarrow$ (2). Suppose (2) holds. By Proposition 2.4, without loss of generality, we assume $\hat{y}(\xi) = [\hat{y}_1(\xi), 0, \ldots, 0]$. Let $f_n := Q_{a,M}^n f$. By assumption in (2), $\lim_{n\to\infty} \|f_n - f_\infty\|_{(W_p^k(\mathbb{R}^s))^{r\times 1}} = 0$ for some $f_\infty \in (W_p^k(\mathbb{R}^s))^{r\times 1}$. When $p = \infty$, we must have $f_\infty \in (C^k(\mathbb{R}^s))^{r\times 1}$ since $f \in (C^k(\mathbb{R}^s))^{r\times 1}$. Let $m := |\det M|$. By induction, $f_n = m^n \sum_{\beta\in\mathbb{Z}^s} a_n(\beta) f(M^n \cdot -\beta)$ for all $n \in \mathbb{N}_0$. Therefore, for $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$, we have

$$\nabla^{\mu,n} f_n = m^n \sum_{\beta\in\mathbb{Z}^s} [\nabla^\mu a_n](\beta) f(M^n \cdot -\beta) \quad \text{with}$$
$$\nabla^{\mu,n} := \nabla_{M^{-n}e_1}^{\mu_1} \cdots \nabla_{M^{-n}e_s}^{\mu_s} \quad \forall n \in \mathbb{N}_0. \tag{4.12}$$

Since the shifts of f are stable, by (4.12), there exists a positive constant C depending only on f such that

$$m^{n-n/p} \|\nabla^\mu a_n\|_{(\ell_p(\mathbb{Z}^s))^{r\times r}} \leqslant C \|\nabla^{\mu,n} f_\infty\|_{(L_p(\mathbb{R}^s))^{r\times 1}} + C \|\nabla^{\mu,n}(f_n - f_\infty)\|_{(L_p(\mathbb{R}^s))^{r\times 1}}.$$

Note that all the functions f_n and f_∞ are supported on $[-L, L]^s$ for some integer L independent of n. Since M is isotropic, there is a constant C_1 independent of n such that

$$\|\nabla^{\mu,n}(f_n - f_\infty)\|_{(L_p(\mathbb{R}^s))^{r\times 1}} \leqslant C_1 m^{-nk/s} \|f_n - f_\infty\|_{(W_p^k(\mathbb{R}^s))^{r\times 1}} \quad \forall |\mu| = k+1.$$

Since $f_\infty \in (W_p^k(\mathbb{R}^s))^{r\times 1}$ or $f_\infty \in (C^k(\mathbb{R}^s))^{r\times 1}$ when $p = \infty$, we deduce that $\lim_{n\to\infty} m^{nk/s} \|\nabla^{\mu,n} f_\infty\|_{(L_p(\mathbb{R}^s))^{r\times 1}} = 0$ for all $|\mu| = k+1$. By assumption $\lim_{n\to\infty} \|f_n - f_\infty\|_{(W_p^k(\mathbb{R}^s))^{r\times 1}} = 0$, we have

$$\lim_{n\to\infty} m^{n(k/s+1-1/p)} \|\nabla^\mu a_n\|_{(\ell_p(\mathbb{Z}^s))^{r\times r}} = 0 \quad \forall |\mu| = k+1,\ \mu \in \mathbb{N}_0^s.$$

Note that $\nabla^\mu a_n = a_n * \nabla^\mu \delta I_r$. Since $[a_n * \nabla^\mu(\delta e_1)](\beta)$ is the first column of $[\nabla^\mu a_n](\beta)$, in particular, we have

$$\lim_{n\to\infty} m^{n(k/s+1-1/p)} \|a_n * \nabla^\mu(\delta e_1)\|_{(\ell_p(\mathbb{Z}^s))^{r\times r}} = 0$$
$$\forall |\mu| = k+1, \mu \in \mathbb{N}_0^s. \tag{4.13}$$

Denote $g := e_1^T f$ to be the first component of f. Since we assume that $\hat{y}(\xi) = [\hat{y}_1(\xi), 0, \ldots, 0]$ and f satisfies the moment conditions of order $k+1$ with respect to y, we have $\hat{g}(0) \neq 0$ and $D^\mu \hat{g}(2\pi\beta) = 0$ for all $|\mu| \leqslant k, \beta \in \mathbb{Z}^s \backslash \{0\}$. Since $f_n \in (W_p^k(\mathbb{R}^s))^{r\times 1}$, by Lemma 4.2, there exists $v_n \in (\ell_0(\mathbb{Z}^s))^{r\times 1}$ such that

$$\left\| f_n - m^n \sum_{\beta\in\mathbb{Z}^s} v_n(\beta) g(M^n \cdot -\beta) \right\|_{(L_p(\mathbb{R}^s))^{r\times 1}} \leqslant C m^{-nk/s} \sum_{|\mu|=k} \omega_p(D^\mu f_n, \rho(M^{-n})), \tag{4.14}$$

where C is a constant independent of f_n and n. Denote

$$g_n := f_n - m^n \sum_{\beta\in\mathbb{Z}^s} v_n(\beta) g(M^n \cdot -\beta) = m^n \sum_{\beta\in\mathbb{Z}^s} (a_n - [v_n, 0, \ldots, 0])(\beta) f(M^n \cdot -\beta).$$

Since the shifts of f are stable, there exists a positive constant C_1 such that

$$\begin{aligned} m^{n(k/s+1-1/p)} \|a_n - [v_n, 0, \ldots, 0]\|_{(\ell_p(\mathbb{Z}^s))^{r\times r}} &\leqslant C_1 m^{nk/s} \|g_n\|_{(L_p(\mathbb{R}^s))^{r\times 1}} \\ &\leqslant CC_1 \sum_{|\mu|=k} \omega_p(D^\mu f_n, \rho(M^{-n})). \end{aligned}$$

By the triangular inequality and the fact $|\mu| \leqslant k$, we have

$$\omega_p(D^\mu f_n, \rho(M^{-n})) \leqslant \omega_p(D^\mu f_\infty, \rho(M^{-n})) + 2\|f_n - f_\infty\|_{(W_p^k(\mathbb{R}^s))^{r\times 1}}.$$

Since $D^\mu f_\infty \in (L_p(\mathbb{R}^s))^{r\times 1}$ (when $p = \infty$, $D^\mu f_\infty \in (C(\mathbb{R}^s))^{r\times 1}$), it follows from the above inequality that $\lim_{n\to\infty} \omega_p(D^\mu f_n, \rho(M^{-n})) = 0$. Consequently,

$$\lim_{n\to\infty} m^{n(k/s+1-1/p)} \|a_n - [v_n, 0, \ldots, 0]\|_{(\ell_p(\mathbb{Z}^s))^{r\times r}} = 0.$$

Since $[a_n * (\delta e_j)](\beta)$ is the jth column of the matrix $(a_n - [v_n, 0, \ldots, 0])(\beta)$ for $j = 2, \ldots, r$, in particular, we have

$$\lim_{n\to\infty} m^{n(k/s+1-1/p)} \|a_n * (\delta e_j)\|_{(\ell_p(\mathbb{Z}^s))^{r\times 1}} = 0 \quad \forall j = 2, \ldots, r. \tag{4.15}$$

Since $\{\nabla^\mu(\delta e_1) \colon |\mu| = k+1\} \cup \{\delta e_j \colon j = 2, \ldots, r\} = \mathscr{B}_{k,y}$ and $\rho(M) = |\det M|^{1/s}$, (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial. By Proposition 4.1, (5) implies that a satisfies the sum rules of order $k+1$ with y. So (5) $\Rightarrow$ (6). By the definition of $\rho(a, M, p)$ and $v_p(a, M)$ in (4.2) and (4.3), it is obvious that (6) $\Rightarrow$ (7) and (7) $\Leftrightarrow$ (8). The equivalence relations between (6), (9) and (10) are standard results on ℓ_p-norm joint spectral radius.

In the following, we show that (6) $\Rightarrow$ (1). Since a satisfies the sum rules of order $k+1$ in (2.8) and (2.9) with y, by $\widehat{Q_{a,M}f}(\xi) = \hat{a}((M^T)^{-1}\xi)\hat{f}((M^T)^{-1}\xi)$, it is easy to verify that $Q_{a,M}f$ also satisfies the moment conditions of order $k+1$ with respect to y. By assumption in (6), there exist two constants $0 < \rho < 1$ and $C > 0$ such that

$$\|a_n * v\|_{(\ell_p(\mathbb{Z}^s))^{r\times 1}} \leqslant C m^{n(1/p-1)} \rho(M)^{-kn} \rho^n \quad \forall v \in \mathscr{B}_{k,y}, n \in \mathbb{N}. \tag{4.16}$$

Let $g := Q_{a,M}f - f$. By Corollary 3.7, we have $[\otimes^k D] \otimes g = \sum_{v \in \mathscr{B}_{k,y}} v * h_{k,v}$ for some compactly supported $h_{k,v} \in (L_p(\mathbb{R}^s))^{1\times s^k}$. Since

$$f_{n+1} - f_n = Q^n_{a,M} g = m^n \sum_{\beta \in \mathbb{Z}^s} a_n(\beta) g(M^n \cdot - \beta),$$

by Proposition 2.1, we have

$$\begin{aligned}[\otimes^k D] \otimes [f_{n+1} - f_n] &= m^n \sum_{\beta \in \mathbb{Z}^s} a_n(\beta)([\otimes^k D] \otimes g)(M^n \cdot - \beta)(\otimes^k M^n) \\ &= m^n \sum_{v \in \mathscr{B}_{k,y}} \sum_{\beta \in \mathbb{Z}^s} [a_n * v](\beta) h_{k,v}(M^n \cdot - \beta)(\otimes^k M^n).\end{aligned}$$

Since $\rho(\otimes^k M^n) = \rho(M)^{kn} < [\rho(M)^k \rho^{-1/2}]^n$ and all $h_{k,v} \in (L_p(\mathbb{R}^s))^{1\times s^k}$ are compactly supported, there exist positive constants C_1 and C_2 such that

$$\begin{aligned}&\|[\otimes^k D] \otimes [f_{n+1} - f_n]\|_{(L_p(\mathbb{R}^s))^{r\times s^k}} \\ &\leqslant C_1 m^n (\rho(M)^k \rho^{-1/2})^n \left\| \sum_{v \in \mathscr{B}_{k,y}} \sum_{\beta \in \mathbb{Z}^s} [a_n * v](\beta) h_{k,v}(M^n \cdot - \beta) \right\|_{(L_p(\mathbb{R}^s))^{r\times s^k}} \\ &\leqslant C_1 C_2 m^{n(1-1/p)} \rho(M)^{kn} \rho^{-n/2} \sum_{v \in \mathscr{B}_{k,y}} \|a_n * v\|_{(\ell_p(\mathbb{Z}^s))^{r\times 1}}.\end{aligned}$$

It follows from (4.16) that

$$\|[\otimes^k D] \otimes [f_{n+1} - f_n]\|_{(L_p(\mathbb{R}^s))^{r\times s^k}} \leqslant C C_1 C_2 (\#\mathscr{B}_{k,y}) \rho^{n/2} \quad \forall n \in \mathbb{N}.$$

Thus, $[\otimes^k D] \otimes f_n$ is a Cauchy sequence in $(L_p(\mathbb{R}^s))^{r\times s^k}$ since $0 < \rho < 1$. Note that all f_n are supported on a fixed compact set. Therefore, we must have

$$\lim_{n\to\infty} \|Q^n_{a,M} f - f_\infty\|_{(W^k_p(\mathbb{R}^s))^{r\times 1}} = \lim_{n\to\infty} \|f_n - f_\infty\|_{(W^k_p(\mathbb{R}^s))^{r\times 1}} = 0$$

for some $f_\infty \in (W^k_p(\mathbb{R}^s))^{r\times 1}$. When (3.1) holds, we must have $\hat{f_\infty} = \hat{\phi}$ and consequently, $f_\infty = \phi$.

Finally, we show (7) $\Rightarrow$ (2). By definition of $\rho(a, M, p)$ in (4.2), we have

$$\rho_J(a, M, p, \tilde{y}) < |\det M|^{1/p-1} \rho(M)^{-k}$$

for some $J \in \mathbb{N}_0$ and $\tilde{y} \in (\ell_0(\mathbb{Z}^s))^{1\times r}$ with $\hat{\tilde{y}}(0) \neq 0$ such that a satisfies the sum rules of order $J+1$ but not $J+2$ in (2.8) and (2.9) with $\tilde{y}$. By Proposition 4.1, $J \geqslant k$ and $\mathscr{V}_{k,\tilde{y}} = \mathscr{V}_{k,y}$. By Proposition 3.5, there exists a function vector $f \in (C^J(\mathbb{R}^s))^{r\times 1}$ such that all the claims in Proposition 3.5 hold with y being replaced by $\tilde{y}$ and $b_\mu = \delta(\mu), |\mu| \leqslant J$. Since $\mathscr{V}_{k,\tilde{y}} = \mathscr{V}_{k,y}$, by Lemma 3.3 and appropriately scaling f by a scalar constant, f must satisfy the moment conditions of order $k+1$ with respect to y. So f is a suitable initial function vector in (2). Since a satisfies the sum rules of order $J+1$ with $\tilde{y}$, it is easy to check that $Q_{a,M}f$ satisfies the moment conditions of order $J+1$ with respect to $\tilde{y}$ and $D^\mu[\hat{\tilde{y}}(\cdot)\widehat{Q_{a,M}f}(\cdot)](0) = \delta(\mu)$ for $|\mu| \leqslant J$. Let

$g = Q_{a,M}f - f$. Now it is easy to verify that for every $|\mu| \leqslant k$, $D^\nu[\hat{\tilde{y}}(\cdot)\hat{g}(\cdot)](2\pi\beta) = 0$ for all $|\nu| \leqslant J$ and $\beta \in \mathbb{Z}^s$. By Theorem 3.6, $D^\mu g = \sum_{v \in \mathscr{B}_{J,\tilde{y}}} v * h_{\mu,v}$, $|\mu| \leqslant k$ for some compactly supported functions $h_{\mu,v} \in L_p(\mathbb{R}^s)$. Now the same argument to show (6) $\Rightarrow$ (1) yields that the sequence $Q^n_{a,M}f$ converges in $(W^k_p(\mathbb{R}^s))^{r \times 1}$. So, (2) holds. $\square$

A comprehensive study of stationary cascade algorithms was given in [3]. For $r = 1$, convergence of scalar cascade algorithms was given by Jia [25] in $L_p(\mathbb{R})$ with $M = 2$, by Han and Jia [21] in $L_p(\mathbb{R}^s)$ with a general dilation matrix, by Lawton et al. [34] in $L_2(\mathbb{R}^s)$ with $M = 2I_s$, by Jia et al. [29] in $W^k_2(\mathbb{R}^s)$ and [19] in $W^k_p(\mathbb{R}^s)$ with an isotropic dilation matrix. For the general case $r > 1$, convergence of vector cascade algorithms was investigated by Jia et al. [31] in $L_p(\mathbb{R})$, by Shen [43] in $L_2(\mathbb{R}^s)$ with $M = 2I_s$, by Goodman and Lee [13] in $W^k_2(\mathbb{R})$, by Micchelli and Sauer [39] in $W^k_p(\mathbb{R})$ and [40] in $W^k_p(\mathbb{R}^s)$, and more recently by Chen et al. [4] in $W^k_p(\mathbb{R}^s)$ with an isotropic dilation matrix (which establishes the equivalence between (1) and (10) in Theorem 4.3), as well as by Li [35,36] and by Zhou [47], and by many other related references in the above papers.

With the help of Propositions 2.1, 2.4 and Corollary 3.7, the proof of Theorem 4.3 is relatively simple and gives us a better picture and understanding of vector cascade algorithms; moreover, we have a clear description of and relation among the set $\mathscr{F}_{k,y,p}$ of initial function vectors, the polynomial space $\mathscr{P}_{k,y}$ and the subspace $\mathscr{V}_{k,y}$ of $(\ell_0(\mathbb{Z}^s))^{r \times 1}$. The statements in (2), (3) and (8) of Theorem 4.3 are new. In particular, (2) settled Q2 in Section 1. The basis $\mathscr{B}_{k,y}$ for the space $\mathscr{V}_{k,y}$ was first introduced here to characterize the convergence of vector cascade algorithms. The characterization in (8) connects the convergence of a cascade algorithm with the smoothness of the refinable function vector and avoids the explicit appearance of the sequence y and the integer k in the quantity $v_p(a, M)$ for the characterization in (8). From the proof of Theorem 4.3, we see that without assuming that M is isotropic, the statements (3)–(10) are equivalent to each other and any one of them implies (1). In fact, in the above proof, (2) $\Rightarrow$ (3) is the only place where we need the assumption that M is isotropic. More technical argument shows that Theorem 4.3 holds when M is a dilation matrix with all its eigenvalues having the same modulus.

The L_p smoothness of a function $f \in L_p(\mathbb{R}^s)$ is measured by its L_p *critical exponent* $v_p(f)$ defined by

$$v_p(f) := \sup\{n + v\colon \|D^\mu f - D^\mu f(\cdot - t)\|_{L_p(\mathbb{R}^s)} \leqslant C_f |t|^v \quad \forall |\mu| = n; t \in \mathbb{R}^s\}.$$

When $f = (f_1, \ldots, f_r)^T$, $v_p(f) := \min\{v_p(f_j)\colon j = 1, \ldots, r\}$. The same proof of Theorem 4.3 to show (2) $\Rightarrow$ (3) and [16, Theorems 3.1 and 3.3] yield that $v_p(\phi) \geqslant v_p(a, M)$. Moreover, when the shifts of ϕ are stable, then one has (see [8] for $p = 2$ and $r = 1$)

$$v_p(a, M) \leqslant v_p(\phi) \leqslant v_p(a, M) \frac{\ln \rho(M)}{\ln \rho(M^{-1})^{-1}}.$$

In particular, when M is isotropic, then $\nu_p(\phi) = \nu_p(a, M)$. For discussion on smoothness of refinable functions and refinable function vectors, see [7,8,11, 16–18,21,26,28,32,33,38,42,44] and many references therein.

In the rest of this section, let us discuss the rate of convergence of a vector cascade algorithm.

Theorem 4.4. *Let M be an $s \times s$ isotropic dilation matrix. Let a be a finitely supported matrix mask on $\mathbb{Z}^s$ with multiplicity r. Let J be the largest integer that is less than $\nu_p(a, M)$; therefore, by Theorem* 4.3, *a satisfies the sum rules of order $J + 1$ in* (2.8) *and* (2.9) *with a sequence $y \in (\ell_0(\mathbb{Z}^s))^{1\times r}$. Let $f \in (W_p^k(\mathbb{R}^s))^{r\times 1}$ satisfy the moment conditions of order $J + 1$ with respect to y and $D^\mu[\hat{y}(\cdot)\hat{f}(\cdot)](0) = \delta(\mu)$ for all $|\mu| \leqslant J - k$ (the existence of such an initial function vector f is guaranteed by Proposition* 3.5). *If $\nu_p(a, M) > k$, then the cascade algorithm associated with mask a, dilation matrix M and the initial function vector f converges in $(W_p^k(\mathbb{R}^s))^{r\times 1}$ and for any $0 < \rho < \rho(M)^{k-\nu_p(a,M)}$, there exists a positive constant C such that*

$$\|Q_{a,M}^n f - \phi\|_{(W_p^k(\mathbb{R}^s))^{r\times 1}} \leqslant C\rho^n,$$
$$\|Q_{a,M}^{n+1} f - Q_{a,M}^n f\|_{(W_p^k(\mathbb{R}^s))^{r\times 1}} \leqslant 2C\rho^n \quad \forall n \in \mathbb{N}, \tag{4.17}$$

where ϕ is the unique M-refinable function vector satisfying $\hat{\phi}(M^T\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$ and $\hat{y}(0)\hat{\phi}(0) = 1$.

Proof. By $\nu_p(a, M) > J$ and Proposition 3.1, (3.4) holds with k being replaced by J. So, by $J \geqslant k$, $D^\mu[\hat{y}(\cdot)\hat{\phi}(\cdot)](0) = \delta(\mu) = D^\mu[\hat{y}(\cdot)\hat{f}(\cdot)](0)$ for all $|\mu| \leqslant J - k$. Now it is easy to check that for all $|\mu| = k$,

$$D^\nu[\hat{y}(\cdot)\widehat{D^\mu[f - \phi]}(\cdot)](2\pi\beta) = 0$$

for all $|\nu| \leqslant J$ and $\beta \in \mathbb{Z}^s$. By Theorem 3.6, for every $|\mu| = k$, we have

$$D^\mu[f - \phi] = \sum_{v \in \mathscr{B}_{J,y}} v * h_{\mu,v}$$

for some compactly supported $h_{\mu,v} \in L_p(\mathbb{R}^s)$. Now the rest of the proof is identical to that of Theorem 4.3 to show (6) $\Rightarrow$ (1). $\square$

5. Refinable Hermite interpolants

As an important family of refinable function vectors, refinable Hermite interpolants are useful in computer-aided geometric design [12,17,23,37,45,47]. In this section, we shall give a simple criterion to characterize a refinable Hermite interpolant in terms of its mask and consequently we settle the question $Q3$ in Section 1.

As a direct consequence of Theorem 4.3, we have the following result which generalizes [21] and was also independently obtained in [6].

Corollary 5.1. *Let M be an $s \times s$ isotropic dilation matrix and a be a finitely supported mask on $\mathbb{Z}^s$ with multiplicity r. Let ϕ be a nonzero compactly supported M-refinable function vector with mask a. If $\phi \in (W_p^k(\mathbb{R}^s))^{r\times 1}$ (when $p = \infty$, we require $\phi \in (C^k(\mathbb{R}^s))^{r\times 1}$) and the shifts of ϕ are stable, then $v_p(a, M) > k$; that is, the vector cascade algorithm associated with mask a and dilation M converges in the Sobolev space $(W_p^k(\mathbb{R}^s))^{r\times 1}$.*

Proof. By Proposition 3.1, (3.4) and (2.8) hold for some $y \in (\ell_0(\mathbb{Z}^s))^{1\times r}$ with $\hat{y}(0) \neq 0$. Item (2) in Theorem 4.3 is satisfied by taking $f = \phi$. The claim follows directly from Theorem 4.3. $\square$

Let us recall the definition of Hermite interpolants given in [17,23]. Let $\Lambda_r := \{\mu \in \mathbb{N}_0^s\colon |\mu| \leqslant r\}$ and by $\#\Lambda_r$ we denote the cardinality of the set Λ_r. Now the elements in Λ_r can be ordered in such a way that $v = (v_1, \ldots, v_s)$ is less that $\mu = (\mu_1, \ldots, \mu_s)$ if either $|v| < |\mu|$ or when $|v| = |\mu|$, $v_j = \mu_j$ for $j = 1, \ldots, i-1$ and $v_i < \mu_i$ for some $1 \leqslant i \leqslant s$. Let $\phi = (\phi_\mu)_{\mu \in \Lambda_r}$ be a column vector of functions on $\mathbb{R}^s$. We say that ϕ is a *Hermite interpolant* of order r if $\phi \in (C^r(\mathbb{R}^s))^{(\#\Lambda_r)\times 1}$ and

$$[D^v \phi_\mu](\alpha) = \delta(\mu - v)\delta(\alpha) \quad \forall \mu, v \in \Lambda_r,\ \alpha \in \mathbb{Z}^s. \tag{5.1}$$

Let $\mathscr{D}^j$ be defined in Proposition 2.1. In other words, (5.1) is equivalent to saying that

$$[1, \mathscr{D}, \mathscr{D}^2, \ldots, \mathscr{D}^r] \otimes \phi(\alpha) = \delta(\alpha) I_{\#\Lambda_r} \quad \forall \alpha \in \mathbb{Z}^s.$$

The definition of a Hermite interpolant can be generalized by replacing Λ_r by a finite subset Λ of $\mathbb{N}_0^s$ such that $0 \leqslant v \leqslant \mu \in \Lambda$ implies $v \in \Lambda$. Note that $(v_1, \ldots, v_s) \leqslant (\mu_1, \ldots, \mu_s)$ means $v_j \leqslant \mu_j$ for all $j = 1, \ldots, s$. This general definition of Hermite interpolants includes the family of tensor product Hermite interpolants.

The following result gives us a simple criterion to characterize a multivariate refinable Hermite interpolant in terms of its mask.

Corollary 5.2. *Let M be an $s \times s$ isotropic dilation matrix and a be a finitely supported mask on $\mathbb{Z}^s$ with multiplicity $\#\Lambda_r$ for some $r \in \mathbb{N}_0$. Let $\phi = (\phi_\mu)_{\mu \in \Lambda_r}$ be a compactly supported M-refinable function vector with mask a and dilation M; that is, $\hat{\phi}(M^T\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$. Then ϕ is a Hermite interpolant of order r if and only if*

(1) $\hat{\phi}_0(0) = 1$ (*this is a normalization condition for a refinable function vector*);
(2) $v_\infty(a, M) > r$ (*In particular, the inequality $v_2(a, M) > r + s/2$ implies $v_\infty(a, M) > r$*);
(3) $a(0) = S(M^{-1}, \Lambda_r)/|\det M|$ *and $a(M\beta) = 0$ for all $\beta \in \mathbb{Z}^s \backslash \{0\}$, where the matrix $S(M^{-1}, \Lambda_r)$ is defined to be*

$$\frac{(M^{-1}x)^\mu}{\mu!} = \sum_{v \in \Lambda_r} S(M^{-1}, \Lambda_r)_{\mu, v} \frac{x^v}{v!}, \quad \mu \in \Lambda_r; \tag{5.2}$$

(4) *The mask a satisfies the sum rules of order* $r+1$ *in* (2.8) *and* (2.9) *with a sequence* $y \in (\ell_0(\mathbb{Z}^s))^{1\times \#\Lambda_r}$ *such that*

$$\frac{(-iD)^{\mu}}{\mu!}\hat{y}(0) = e_{\mu}^T, \quad |\mu| \leqslant r, \ \ \mu \in \mathbb{N}_0^s, \tag{5.3}$$

where e_{μ} *denotes the* μ*th coordinate unit vector in* $\mathbb{R}^{(\#\Lambda_r)}$.

Proof. Let $m := |\det M|$. Suppose that ϕ is a Hermite interpolant of order r. Then $\phi \in (C^r(\mathbb{R}^s))^{(\#\Lambda_r)\times 1}$ and the shifts of ϕ are stable (in fact, linearly independent). By Theorem 4.3, $\nu_\infty(a, M) > r$. So, (2) holds. From the refinement equation $\phi(M^{-1}\cdot) = m\sum_{\beta\in\mathbb{Z}^s} a(\beta)\phi(\cdot-\beta)$, by Proposition 2.1, for $j = 0, \ldots, r$, we have

$$[\mathscr{D}^j \otimes \phi](M^{-1}\cdot)S(M^{-1}, O_j) = \mathscr{D}^j \otimes [\phi(M^{-1}\cdot)] = m\sum_{\beta\in\mathbb{Z}^s} a(\beta)[\mathscr{D}^j \otimes \phi](\cdot-\beta).$$

Note that $S(M^{-1}, \Lambda_r) = \operatorname{diag}(S(M^{-1}, O_0), S(M^{-1}, O_1), \ldots, S(M^{-1}, O_r))$. It follows from the definition of a Hermite interpolant of order r in (5.1) that for any $\alpha \in \mathbb{Z}^s$,

$$\begin{aligned}\delta(\alpha)S(M^{-1}, \Lambda_r)/m &= [1, \mathscr{D}, \ldots, \mathscr{D}^r] \otimes \phi(\alpha)S(M^{-1}, \Lambda_r)/m \\ &= \sum_{\beta\in\mathbb{Z}^s} a(\beta)[1, \mathscr{D}, \ldots, \mathscr{D}^r] \otimes \phi(M\alpha-\beta) = a(M\alpha).\end{aligned}$$

So (3) holds. Since ϕ is a refinable Hermite interpolant, by Proposition 3.2 and Theorem 4.3, we must have $(p * y) * \phi = p$ for all $p \in \Pi_r$ and a satisfies the sum rules of order $r+1$ in (2.8) and (2.9) for some sequence $y \in (\ell_0(\mathbb{Z}^s))^{1\times\#\Lambda_r}$. By (2.13),

$$(p * y) * \phi = \sum_{\mu\in\Lambda_r} D^{\mu}p(\beta)\frac{(-iD)^{\mu}}{\mu!}\hat{y}(0)\phi(\cdot-\beta) = p \quad \forall p \in \Pi_r. \tag{5.4}$$

Since ϕ is a Hermite interpolant, by (5.4), for all $p \in \Pi_r$ and $\alpha \in \mathbb{Z}^s$, we have

$$\begin{aligned}[1, \mathscr{D}, \ldots, \mathscr{D}^r] \otimes p(\alpha) &= \sum_{\mu\in\Lambda_r} D^{\mu}p(\beta)\frac{(-iD)^{\mu}}{\mu!}\hat{y}(0)[1, \mathscr{D}, \ldots, \mathscr{D}^r] \otimes \phi(\alpha-\beta) \\ &= \sum_{\mu\in\Lambda_r} D^{\mu}p(\alpha)\frac{(-iD)^{\mu}}{\mu!}\hat{y}(0).\end{aligned}$$

It follows from the above identity that the sequence y must satisfy (5.3). So, (4) holds. In particular, from (5.4), we have $\sum_{\beta\in\mathbb{Z}^s} \phi_0(\cdot-\beta) = 1$ and therefore, $\hat{\phi}_0(0) = 1$. So, (1) holds.

Conversely, it is known [17] that there is a 2-refinable function vector $\psi \in (C^r(\mathbb{R}))^{(r+1)\times 1}$ which is a Hermite interpolant of order r whose mask is supported on $[-1, 1]$. Such ψ is in fact a B-spline function vector with multiple knots. Define a function vector f by

$$f_{(\mu_1,\ldots,\mu_s)}(t_1, \ldots, t_s) := \psi_{\mu_1}(t_1)\cdots\psi_{\mu_s}(t_s), \quad (\mu_1, \ldots, \mu_s) \in \Lambda_r.$$

It is easy to verify that f is a Hermite interpolant of order r and (5.4) holds with ϕ being replaced by f and with the sequence y in (4). So f satisfies the moment conditions of order $r+1$ with respect to y. Note that (1) implies $\hat{y}(0)\hat{\phi}(0) = \hat{\phi}_0(0) = 1$. By Theorem 4.3, it follows from (2) and the fact $\hat{y}(0)\hat{\phi}(0) = \hat{y}(0)\hat{f}(0) = 1$ that the cascade algorithm associated with mask a, dilation matrix M and the initial function vector f converges to ϕ in $(C^r(\mathbb{R}^s))^{(\#\Lambda_r)\times 1}$. Let $f_n := Q^n_{a,M}f$. By Proposition 2.1 and $f_n(M^{-n}\cdot) = m^n \sum_{\beta\in\mathbb{Z}^s} a_n(\beta)f(\cdot-\beta)$, since f is a Hermite interpolant, we observe that

$$\begin{aligned}([1,\mathscr{D},\dots,\mathscr{D}^r]\otimes f_n)(\alpha)S(M^{-n},\Lambda_r) &= m^n \sum_{\beta\in\mathbb{Z}^s}[1,\mathscr{D},\dots,\mathscr{D}^r]\otimes f(M^n\alpha-\beta)\\ &= m^n a_n(M^n\alpha).\end{aligned}$$

By $a(M\beta) = \delta(\beta)S(M^{-1},\Lambda_r)/m$ for all $\beta\in\mathbb{Z}^s$, by induction we see that $a_n(M^n\beta) = \delta(\beta)\,[S(M^{-1},\Lambda_r)]^n/m^n$ for all $\beta\in\mathbb{Z}^s$. Observing $S(M^{-n},\Lambda_r) = [S(M^{-1},\Lambda_r)]^n$, we conclude that $[1,\mathscr{D},\dots,\mathscr{D}^r]\otimes f_n(\alpha) = \delta(\alpha)I_{\#\Lambda_r}$ and therefore, f_n must be a Hermite interpolant for all n. Consequently, ϕ must be a Hermite interpolant of order r since $D^\nu\phi_\mu(\alpha) = \lim_{n\to\infty} D^\nu[f_n]_\mu(\alpha) = \delta(\mu-\nu)\delta(\alpha)$ for all $\mu,\nu\in\Lambda_r$ and $\alpha\in\mathbb{Z}^s$. $\square$

Univariate refinable Hermite interpolants have been studied in [12,17,37,45,46]. We say that a mask a is a *Hermite interpolatory mask* of order r with respect to the dilation matrix M if (3) and (4) in Corollary 5.2 hold. The concept of Hermite interpolatory masks in the univariate setting has been introduced in [17] and a family of Hermite interpolatory masks of order r with a general dilation factor has been constructed in [17].

In the univariate setting with $M=2$, a necessary and sufficient condition for a refinable function vector to be a Hermite interpolant was obtained in [46]. Our characterization in Corollary 5.2 is much simpler than that of [46] even for the univariate case. Refinable Hermite interpolants have been also discussed in [23]. See [23] for construction of multivariate Hermite interpolatory masks with symmetry.

In the rest of this section, we have the result about the sequence y for a Hermite interpolatory mask.

Proposition 5.3. *Let M be an $s\times s$ dilation matrix. Let a be a finitely supported mask on $\mathbb{Z}^s$ with multiplicity $\#\Lambda_r$ such that $a(0) = S(M^{-1},\Lambda_r)/|\det M|$ and $a(M\beta)=0$ for all $\beta\in\mathbb{Z}^s\backslash\{0\}$. Suppose that a satisfies the sum rules of order $k+1$ $(k\geqslant r)$ in* (2.8) *and* (2.9) *with some sequence $y\in(\ell_0(\mathbb{Z}^s))^{1\times r}$. Let $\sigma=(\sigma_1,\dots,\sigma_s)$, where $\sigma_1,\dots,\sigma_s$ are all the eigenvalues of M. If $\sigma^\mu\notin\{\sigma^\nu:\ \nu\in\Lambda_r\}$ for all $r<|\mu|\leqslant k$ and $\mu\in\mathbb{N}_0^s$ (this clearly holds when M is an isotropic dilation matrix*), *then we must have $D^\mu\hat{y}(0)=0$ for all $r<|\mu|\leqslant k$.*

Proof. Denote $y_\mu := (-iD)^\mu \hat{y}(0)/\mu!$ and $J^a_\alpha(\mu) := |\det M| \sum_{\beta\in\mathbb{Z}^s} a(\alpha + M\beta)(M^{-1}\alpha + \beta)^\mu/\mu!$. Using the Leibniz differentiation formula, the definition of sum rules in (2.8) and (2.9) can be equivalently translated into [17]

$$\sum_{0\leqslant\nu\leqslant\mu} (-1)^{|\nu|} y_{\mu-\nu} J^a_\alpha(\nu) = \sum_{|\nu|=|\mu|} S(M^{-1}, O_{|\mu|})_{\mu,\nu} y_\nu \quad \forall \mu\in\Lambda_k,\ \alpha\in\mathbb{Z}^s. \tag{5.5}$$

Since $a(M\beta) = \delta(\beta)S(M^{-1}, \Lambda_r)/|\det M|$ for all $\beta\in\mathbb{Z}^s$, we have $J^a_0(\mu) = \delta(\mu)S(M^{-1}, \Lambda_r)$. Setting $\alpha = 0$ in (5.5), we deduce that $y_\mu S(M^{-1}, \Lambda_r) = y_\mu J^a_0(0) = \sum_{|\nu|=|\mu|} S(M^{-1}, O_{|\mu|})_{\mu,\nu} y_\nu$, $\mu\in\Lambda_k$. Denote $Y_n := (y_\mu)_{\mu\in O_n}$ as a $\#O_n \times \#\Lambda_r$ matrix. Therefore, we have $Y_n S(M^{-1}, \Lambda_r) = S(M^{-1}, O_n)\, Y_n$ for $n = 0, \ldots, k$ which is equivalent to $(S(M^{-1}, \Lambda_r)^T \otimes I_{\#O_n})\mathrm{vec}(Y_n) = (I_{\#\Lambda_r} \otimes S(M^{-1}, O_n))\mathrm{vec}(Y_n)$ for all $n = 0, \ldots, k$. By assumption on M, we see that

$$\begin{aligned} &S(M^{-1}, \Lambda_r)^T \otimes I_{\#O_n} - I_{\#\Lambda_r} \otimes S(M^{-1}, O_n) \\ &\quad = [S(M^{-1}, \Lambda_r)^T \otimes S(M, O_n) - I_{\#\Lambda_k} \otimes I_{\#O_n}][I_{\#\Lambda_k} \otimes S(M^{-1}, O_n)] \end{aligned}$$

is invertible for all $n = r+1, \ldots, k$. So, we have $Y_n = 0$ for all $n = r+1, \ldots, k$ which completes the proof. $\square$

If a is a Hermite interpolatory mask of order r with respect to a dilation matrix M and a satisfies the sum rules of order k ($k\geqslant r$) with a sequence y, then (5.3) holds and $(-iD)^\mu \hat{y}(0) = 0$ for all $r<|\mu|\leqslant k$; moreover, $\mathscr{P}_{k,y} = \{(D^\mu p)_{\mu\in\Lambda_r}\colon p\in\Pi_k\}$ and by Proposition 2.10, $S_{a,M}((D^\mu p)_{\mu\in\Lambda_r}) = (D^\mu[p(M^{-1}\cdot)])_{\mu\in\Lambda_r}$ for all $p\in\Pi_k$.

6. Error estimate of vector cascade algorithms in Sobolev spaces

In applications, when the coefficients of a mask (such as the Daubechies' orthogonal masks in [9]) are irrational numbers, one often needs to truncate such a mask. Heil and Collela [24] studied how such truncation affects a scalar refinable function in the univariate L_∞ case. Daubechies and Huang [10] studied how truncation affects the associated scalar refinable function in the univariate L_∞ case in the frequency domain. Han [14,15] first provided a sharp error estimate for multivariate scalar refinable functions and for their cascade algorithms with a perturbed mask in any L_p norm. More specifically, it was proved in [14,15] that if a scalar cascade algorithm associated with a mask a converges in the L_p norm, then there exist two positive constants η and C such that for any mask b such that $\|a-b\|_{\ell_1(\mathbb{Z}^s)}<\eta$ and b satisfies the sum rules of order 1, one has

$$\|Q^n_{a,M} f - Q^n_{b,M} f\|_{L_p(\mathbb{R}^s)} \leqslant C\|a-b\|_{\ell_1(\mathbb{Z}^s)} \quad \forall n\in\mathbb{N}$$

and

$$\|\phi^a - \phi^b\|_{L_p(\mathbb{R}^s)} \leqslant C\|a-b\|_{\ell_1(\mathbb{Z}^s)},$$

where f is an initial function in the scalar cascade algorithm, and ϕ^a and ϕ^b denote the scalar M-refinable functions with masks a and b, respectively, with the standard normalization condition $\hat{\phi}^a(0) = \hat{\phi}^b(0) = 1$.

The main idea in [14,15] was used in [20] to obtain error estimate for vector cascade algorithms in the univariate L_p case, and recently was generalized by Chen and Plonka [5] to establish error estimates for scalar cascade algorithms in a Sobolev space with a particular initial function which is the tensor product of a certain B-spline function. Such a restriction on the initial functions in [5] was completely removed in [19].

As we shall discuss later, the situation for vector cascade algorithms is much more complicated. Let a be a finitely supported mask on $\mathbb{Z}^s$ with multiplicity r. We denote by $y^a \in (\ell_0(\mathbb{Z}^s))^{1\times r}$ a sequence such that

$$D^\mu[\hat{y}^a(M^T\cdot)\hat{a}(\cdot)](0) = D^\mu \hat{y}^a(0) \quad \forall |\mu| \leqslant k \quad \text{and} \quad \hat{y}^a(0) \neq 0. \tag{6.1}$$

Note that there are many choices for such a sequence y^a. But when (3.1) holds, by Proposition 3.2, up to a scalar multiplication, there exists a unique sequence $y^a \in (\ell(\Lambda_k))^{1\times r}$, where $\Lambda_k := \{\beta \in \mathbb{N}_0^s\colon\ |\beta| \leqslant k\}$.

In the scalar case $r = 1$, by uniformly normalizing y^a by $\hat{y}^a(0) = 1$, we observe that the set $\mathscr{F}_{k,y^a,p}$ is independent of the mask a since $\mathscr{F}_{k,y^a,p} = \mathscr{F}_{k,\delta,p}$. However, when $r > 1$, it is not easy to uniformly normalize the sequence y^a and in fact $\mathscr{F}_{k,y^a,p}$ indeed depends on the sequence y^a which in turn depends on the mask a. Such difficulty makes the error estimate in the vector case much more complicated. As a matter of fact, the error estimate for the univariate vector cascade algorithms in [20] requires that the perturbed mask satisfy a strict condition which makes such an error estimate in [20] less useful in practice. It is the purpose of this section to satisfactorily settle Q4 in Section 1 for the vector case in any dimension using the results in previous sections.

Lemma 6.1. *Let M be an $s \times s$ isotropic dilation matrix. Let k be a nonnegative integer and Ω be a compact subset of $\mathbb{Z}^s$ with $0 \in \Omega$. Let a be a finitely supported matrix mask on $\mathbb{Z}^s$ with multiplicity r. We assume that*

(a) *$a(\beta) = 0$ for all $\beta \in \mathbb{Z}^s \backslash \Omega$; that is, $a \in (\ell(\Omega))^{r\times r}$;*
(b) *1 is a simple eigenvalue of $\hat{a}(0)$ and all other eigenvalues of $\hat{a}(0)$ are less than $\rho(M)^{-k}$ in modulus;*
(c) *a satisfies the sum rules of order $k+1$ in* (2.8) *and* (2.9) *with a sequence $y^a \in (\ell_0(\mathbb{Z}^s))^{1\times r}$ and $\hat{y}^a(0) \neq 0$;*
(d) *$\rho_k(a, M, p, y^a) < |\det M|^{1/p-1}\rho(M)^{-k}$; that is, the cascade algorithm associated with mask a, dilation M and every initial function vector $f \in \mathscr{F}_{k,y^a,p}$ converges in the Sobolev space $(W_p^k(\mathbb{R}^s))^{r\times 1}$.*

Note that (d) *implies both* (b) *and* (c) *by Theorem* 4.3. *Then there exist positive constants η and C such that for every $b \in N_\eta(a, k, M, \Omega)$ satisfying $\mathscr{V}_{k,y^b} = \mathscr{V}_{k,y^a}$, one* *has* $\rho_k(b, M, p, y^b) < |\det M|^{1/p-1}\rho(M)^{-k}$ *and*

$$\|a_n * v - b_n * v\|_{(\ell_p(\mathbb{Z}^s))^{r\times 1}} \leqslant C|\det M|^{n(1/p-1)}\rho(M)^{-kn}\|a-b\|_{(\ell_1(\mathbb{Z}^s))^{r\times r}}$$

$$\forall v \in \mathscr{B}_{k-1,y^a}, \quad n \in \mathbb{N}, \tag{6.2}$$

where a_n *is defined in* (2.7) *and* $\hat{b_n}(\xi) = \hat{b}((M^T)^{n-1}\xi)\cdots\hat{b}(M^T\xi)\hat{b}(\xi)$. *By* $b \in N_\eta(a, k, M, \Omega)$ *we mean*

(1) $b(\beta) = 0$ *for all* $\beta \in \mathbb{Z}^s\backslash\Omega$; *that is*, $b \in (\ell(\Omega))^{r\times r}$;
(2) $\|a-b\|_{(\ell_1(\mathbb{Z}^s))^{r\times r}} < \eta$;
(3) 1 *is a simple eigenvalue of* $\hat{b}(0)$ *and all other eigenvalues of* $\hat{b}(0)$ *are less than* $\rho(M)^{-k}$ *in modulus*;
(4) b *satisfies the sum rules of order* $k+1$ *in* (2.8) *and* (2.9) *with a sequence* $y^b \in (\ell_0(\mathbb{Z}^s))^{1\times r}$.

Note that both η *and* C *are independent of* b *and* n.

Proof. Denote $m := |\det M|$. Let A_ε and B_ε be defined in (4.5) for the masks a and b with K defined in (10) of Theorem 4.3 and $K_0 := \Omega - \Gamma_M + \{\alpha \in \mathbb{Z}^s\colon |\alpha| \leqslant 1\}$, respectively. Denote $\mathscr{A} := \{A_\varepsilon|_{\mathscr{V}_{k,y^a} \cap (\ell(K))^{r\times 1}}\colon \varepsilon \in \Gamma_M\}$ and $\mathscr{B} := \{B_\varepsilon|_{\mathscr{V}_{k,y^a} \cap (\ell(K))^{r\times 1}}\colon \varepsilon \in \Gamma_M\}$. As in [5,14,15,19], there exist $\eta > 0$, $0 < \rho < 1$ and $C_1 > 0$ such that for all $b \in N_\eta(a, k, M, \Omega)$ such that $\mathscr{V}_{k,y^b} = \mathscr{V}_{k,y^a}$, one has $\|\mathscr{B}^n\|_p \leqslant C_1 m^{n/p-n}\rho(M)^{-kn}\rho^n$ for all $n \in \mathbb{N}$.

By Proposition 2.4, we assume that $\hat{y^a}(\xi) = [\hat{y}_1^a(\xi), 0, \ldots, 0]$ with $\hat{y}_1^a(0) = 1$ and by (c), $\hat{a}(\xi)$ must take the form of (2.10) and (2.11). Now by Lemma 3.3 and (3), we see that $\mathscr{V}_{k,y^b} = \mathscr{V}_{k,y^a}$ if and only if $\hat{b}(\xi)$ also takes the form of (2.10) and (2.11) with a being replaced by b. We observe [5,14,15,19] that for all $v \in (\ell_0(\mathbb{Z}^s))^{r\times 1}$,

$$\|b_n * v - a_n * v\|_{(\ell_p(\mathbb{Z}^s))^{r\times 1}}$$

$$\leqslant \sum_{j=1}^{n}\left(\sum_{\varepsilon_1,\ldots,\varepsilon_n \in \Gamma_M} \|B_{\varepsilon_1}\cdots B_{\varepsilon_{j-1}}(B_{\varepsilon_j} - A_{\varepsilon_j})A_{\varepsilon_{j+1}}\cdots A_{\varepsilon_n}v\|^p_{(\ell_p(\mathbb{Z}^s))^{r\times 1}}\right)^{1/p}.$$

Note that $\mathscr{V}_{k,y^a} = \mathscr{V}_{k,\delta} \times (\ell_0(\mathbb{Z}^s))^{(r-1)\times 1}$. Using the special form of $\hat{a}(\xi)$ and $\hat{b}(\xi)$ in (2.10) and (2.11), by (2.19) and (2.20), we see that $(B_{\varepsilon_j} - A_{\varepsilon_j})\mathscr{V}_{k-1,y^a} \subseteq \mathscr{V}_{k,y^a}$. By Proposition 2.5, $A_{\varepsilon_{j+1}}\cdots A_{\varepsilon_n}\mathscr{V}_{k-1,y^a} \subseteq \mathscr{V}_{k-1,y^a}$. Note that $\|B_{\varepsilon_j} - A_{\varepsilon_j}\| \leqslant \|a-b\|_{(\ell_1(\mathbb{Z}^s))^{r\times r}}$. Now by a similar argument as in [5], for $v\in\mathscr{B}_{k-1,y^a}$, one has

$$\begin{aligned}
&\sum_{\varepsilon_{j+1},\dots,\varepsilon_n\in\Gamma_M}\ \sum_{\varepsilon_1,\dots,\varepsilon_j\in\Gamma_M} ||B_{\varepsilon_1}\cdots B_{\varepsilon_{j-1}}(B_{\varepsilon_j}-A_{\varepsilon_j})A_{\varepsilon_{j+1}}\cdots A_{\varepsilon_n}v||^p_{(\ell_p(\mathbb{Z}^s))^{r\times 1}}\\
&\quad\leqslant C_1 m^{(j-1)(1-p)}\rho(M)^{-k(j-1)p}\rho^{(j-1)p}||B_{\varepsilon_j}-A_{\varepsilon_j}||^p\\
&\quad\times\sum_{\varepsilon_{j+1},\dots,\varepsilon_n\in\Gamma_M}||A_{\varepsilon_{j+1}}\cdots A_{\varepsilon_n}v||^p_{(\ell_p(\mathbb{Z}^s))^{r\times 1}}\\
&\quad\leqslant C_2||v||^p_{(\ell_1(\mathbb{Z}^s))^{r\times 1}}\rho^{(j-1)p}||a-b||^p_{(\ell_1(\mathbb{Z}^s))^{r\times r}}m^{n(1-p)}\rho(M)^{-knp}
\end{aligned}$$

for some constants C_1 and C_2 independent of b and n. It follows from the above inequalities that

$$\begin{aligned}
&||b_n*v-a_n*v||_{(\ell_p(\mathbb{Z}^s))^{r\times 1}}\\
&\quad\leqslant C_2^{1/p}||v||_{(\ell_1(\mathbb{Z}^s))^{r\times 1}}m^{n(1-1/p)}\rho(M)^{-kn}||a-b||_{(\ell_1(\mathbb{Z}^s))^{r\times r}}\sum_{j=1}^{n}\rho^{j-1}.
\end{aligned}$$

So, (6.2) holds with C given by

$$C=C_2^{1/p}\max\{||v||_{(\ell_1(\mathbb{Z}^s))^{r\times 1}}\colon v\in\mathscr{B}_{k-1,y^a}\}\sum_{j=0}^{\infty}\rho^j<\infty. \qquad\square$$

The following is the main result in this section which settles Q4 in Section 1.

Theorem 6.2. *Let M be an $s\times s$ isotropic dilation matrix. Under the same assumptions* (a)–(d) *on the mask a as in Lemma* 6.1. *Then there exist positive constants η, C_1, C_2 and C_3 such that*

(1) *For every initial function vector $f\in\mathscr{F}_{k,y^a,p}$,*

$$\begin{aligned}
&||Q^n_{a,M}f-Q^n_{b,M}f\,||_{(W^k_p(\mathbb{R}^s))^{r\times 1}}\\
&\quad\leqslant C_1||a-b||_{(\ell_1(\mathbb{Z}^s))^{r\times r}}\quad\forall n\in\mathbb{N}, b\in N_\eta(a,k,M,\Omega)
\end{aligned}\tag{6.3}$$

provided that $\mathscr{F}_{k,y^b,p}=\mathscr{F}_{k,y^a,p}$;

(2) *With an appropriate choice for y^b, we have*

$$||y^a-y^b||_{(\ell_1(\mathbb{Z}^s))^{1\times r}}\leqslant C_2||a-b||_{(\ell_1(\mathbb{Z}^s))^{r\times r}}\quad\forall b\in N_\eta(a,k,M,\Omega);\tag{6.4}$$

(3) $\rho_k(b,M,p,y^b)<|\det M|^{1/p-1}\rho(M)^{-k}$; *that is, for every mask $b\in N_\eta(a,k,M,\Omega)$, the cascade algorithm associated with mask b, dilation M and every $f\in\mathscr{F}_{k,y^b,p}$ converges in the Sobolev space $(W^k_p(\mathbb{R}^s))^{r\times 1}$;*

(4) *With the choice y^b in* (2), *let ϕ^a and ϕ^b be two compactly supported M-refinable function vectors such that*

$$\hat\phi^a(M^T\xi)=\hat a(\xi)\hat\phi^a(\xi),\quad \hat y^a(0)\hat\phi^a(0)=1$$

and

$$\hat{\phi}^b(M^T\xi) = \hat{b}(\xi)\hat{\phi}^b(\xi), \quad \hat{y}^b(0)\hat{\phi}^b(0) = 1. \tag{6.5}$$

Then ϕ^a *and* ϕ^b *belong to the Sobolev space* $(W_p^k(\mathbb{R}^s))^{r\times 1}$ *and one has the following estimate*:

$$\|\phi^a - \phi^b\|_{(W_p^k(\mathbb{R}^s))^{r\times 1}} \leqslant C_3\|a-b\|_{(\ell_1(\mathbb{Z}^s))^{r\times r}} \quad \forall b\in N_\eta(a,k,M,\Omega). \tag{6.6}$$

Note that all η, C_1, C_2 *and* C_3 *are independent of n and b.*

Proof. By Lemma 3.3, $\mathscr{F}_{k,y^a,p} = \mathscr{F}_{k,y^b,p}$ implies $\mathscr{V}_{k,y^a} = \mathscr{V}_{k,y^b}$. Since $f\in\mathscr{F}_{k,y^a,p}$, by Corollary 3.7, $[\otimes^k D]\otimes f = \sum_{v\in\mathscr{B}_{k-1,y^a}} v * H_{k,v}$ for some compactly supported function vectors $H_{k,v}\in(L_p(\mathbb{R}^s))^{1\times s^k}$. Let $m := |\det M|$. By induction,

$$Q_{a,M}^n f - Q_{b,M}^n f = m^n \sum_{\beta\in\mathbb{Z}^s} (a_n - b_n)(\beta) f(M^n\cdot-\beta).$$

Therefore, by Proposition 2.1, we have

$$\begin{aligned}
&[\otimes^k D]\otimes[Q_{a,M}^n f - Q_{b,M}^n f]\\
&\quad = m^n \sum_{\beta\in\mathbb{Z}^s} (a_n - b_n)(\beta)([\otimes^k D]\otimes f)(M^n\cdot-\beta)(\otimes^k M^n)\\
&\quad = m^n \sum_{v\in\mathscr{B}_{k-1,y^a}} \sum_{\beta\in\mathbb{Z}^s} (a_n * v - b_n * v)(\beta) H_{k,v}(M^n\cdot-\beta)(\otimes^k M^n).
\end{aligned}$$

Since $\rho(\otimes^k M^n) = \rho(M)^{kn}$ and all $H_{k,v}$ are compactly supported function vectors in $(L_p(\mathbb{R}^s))^{1\times s^k}$, there exists a positive constant C_0 such that

$$\begin{aligned}
&\|[\otimes^k D]\otimes[Q_{a,M}^n f - Q_{b,M}^n f]\|_{(L_p(\mathbb{R}^s))^{r\times s^k}}\\
&\quad \leqslant C_0 m^{n(1-1/p)}\rho(M)^{kn} \sum_{v\in\mathbf{B}_{k-1,y^a}} \|a_n * v - b_n * v\|_{(\ell_p(\mathbb{Z}^s))^{r\times 1}}.
\end{aligned}$$

By Lemma 6.1, (6.2) holds. Consequently, we deduce that

$$\|[\otimes^k D]\otimes[Q_{a,M}^n f - Q_{b,M}^n f]\|_{(L_p(\mathbb{R}^s))^{r\times s^k}} \leqslant CC_0(\#\mathscr{B}_{k-1,y^a})\|a-b\|_{(\ell_1(\mathbb{Z}^s))^{r\times r}}$$

for all $n\in\mathbb{N}$ and $b\in N_\eta(a,k,M,\Omega)$ such that $\mathscr{V}_{k,y^b} = \mathscr{V}_{k,y^a}$. Since all $Q_{a,M}^n f$ and $Q_{b,M}^n f$ are supported on a fixed compact set, we conclude that (6.3) holds for some constant C_1 independent of b and n. In fact, (6.3) holds for every $f\in\mathscr{F}_{k,y^a,p}\cap\mathscr{F}_{k,y^b,p}$.

In the following, we prove (2)–(4). By Proposition 2.4, without loss of generality, we assume that $\hat{y}^a(\xi) = [\hat{y}_1^a(\xi), 0, \ldots, 0]$ with $\hat{y}_1^a(0) = 1$. Write $\hat{y}^b(\xi) = [\hat{y}_1^b(\xi), \hat{y}_2^b(\xi)]$. Since $b\in N_\eta(a,k,M,\Omega)$ implies that 1 is a simple eigenvalue of $\hat{b}(0)$, there is a unique solution $\hat{y}^b(0)$ to the equation $\hat{y}^b(0)\hat{b}(0) = \hat{y}^b(0)$ with $\hat{y}_1^b(0) = 1$. In fact, $\hat{y}_2^b(0) = \hat{y}_1^b(0)\hat{b}_{1,2}(0)[I_{r-1} - \hat{b}_{2,2}(0)]^{-1}$. Since $\hat{a}(\xi)$ takes the form of (2.10), we have

$\rho(\hat{a}_{2,2}(0)) < 1$ and $\hat{a}_{1,2}(0) = 0$. Therefore, $\hat{y}_2^b(0)$ is well defined since $\rho(\hat{b}_{2,2}(0)) < 1$ when η is small enough. Since $\hat{a}_{1,2}(0) = 0$, by condition (3) in Lemma 6.1,

$$\begin{aligned}||\hat{y}^a(0) - \hat{y}^b(0)|| &= ||\hat{y}_2^b(0)|| \leqslant C||\hat{b}_{1,2}(0)|| \\ &= C||\hat{a}_{1,2}(0) - \hat{b}_{1,2}(0)|| \leqslant C||a-b||_{(\ell_1(\mathbb{Z}^s))^{r\times r}}\end{aligned}$$

for some constant C. By condition (3) in Lemma 6.1, it follows from Lemma 2.2 that there is a unique solution $\{D^\mu \hat{y}^b(0)\colon 0 < |\mu| \leqslant k\}$ to the system of linear equations $D^\mu[\hat{y}^b(M^T\cdot)\hat{b}(\cdot)](0) = D^\mu \hat{y}^b(0)$ for $0 < |\mu| \leqslant k$ and in fact it is easy to check that there exists a positive constant C_0 independent of b such that

$$\sup_{\mu\in\Lambda_k} ||D^\mu \hat{y}^b(0) - D^\mu \hat{y}^a(0)|| \leqslant C_0||a-b||_{(\ell_1(\mathbb{Z}^s))^{r\times r}} \quad \forall b \in N_\eta(a,k,M,\Omega). \tag{6.7}$$

It is well known that there exists a unique $y \in (\ell(\Lambda_k))^{1\times r}$ such that $D^\mu \hat{y}(0)$, $|\mu| \leqslant k$ are preassigned. Choose the sequences y^a and y^b in $(\ell(\Lambda_k))^{1\times r}$. It follows from (6.7) that (6.4) holds for some constant C_2 independent of b.

Let y^b be the sequence chosen above with $\hat{y}_1^b(0) = 1$. As in the proof of Proposition 2.4, there exists a unique sequence $c \in (\ell(\Lambda_k))^{1\times(r-1)}$ such that $D^\mu \hat{c}(0) = D^\mu[\hat{y}_2^b(\cdot)/\hat{y}_1^b(\cdot)](0)$ for all $|\mu| \leqslant k$. So,

$$\sup_{\mu\in\Lambda_k} ||D^\mu \hat{c}(0)|| \leqslant C_1 \sup_{\mu\in\Lambda_k} ||D^\mu \hat{y}_2^b(0)|| \leqslant C_1 \sup_{\mu\in\Lambda_k} ||D^\mu \hat{y}^a(0) - D^\mu \hat{y}^b(0)||$$

for some constant C_1. Consequently, by (6.7),

$$||c||_{(\ell_1(\mathbb{Z}^s))^{1\times(r-1)}} \leqslant C \sup_{\mu\in\Lambda_k} ||D^\mu \hat{y}^b(0) - D^\mu \hat{y}^a(0)|| \leqslant CC_0||a-b||_{(\ell_1(\mathbb{Z}^s))^{r\times r}} \tag{6.8}$$

for some constant C independent of b. Define $U \in (\ell_0(\mathbb{Z}^s))^{r\times r}$ by

$$\hat{U}(\xi) := \begin{bmatrix} 1 & -\hat{c}(\xi) \\ 0 & I_{r-1} \end{bmatrix}.$$

Define

$$\hat{\tilde{b}}(\xi) := \hat{U}(M^T\xi)^{-1}\hat{b}(\xi)\hat{U}(\xi), \quad \hat{y^{\tilde{b}}}(\xi) = \hat{y}^b(\xi)\hat{U}(\xi) \quad \text{and} \quad \hat{\phi}^{\tilde{b}}(\xi) := \hat{U}(\xi)^{-1}\hat{\phi}^b(\xi),$$

where ϕ^b is given in (6.5). Then $\hat{\phi}^{\tilde{b}}(M^T\xi) = \hat{\tilde{b}}(\xi)\hat{\phi}^{\tilde{b}}(\xi)$ and $\hat{y}(0)\hat{\phi}^{\tilde{b}}(0) = 1$. It follows from (6.8) that there exists a small enough $\eta' > 0$ such that $b \in N_{\eta'}(a,k,M,\Omega)$ implies $\tilde{b} \in N_\eta(a,k,M,\Omega)$ since

$$||U - I_r\delta||_{(\ell_1(\mathbb{Z}^s))^{r\times r}} = ||c||_{(\ell_1(\mathbb{Z}^s))^{1\times(r-1)}} \leqslant CC_0||a-b||_{(\ell_1(\mathbb{Z}^s))^{r\times r}}. \tag{6.9}$$

Since $D^\mu \hat{y}_j^a(0) = D^\mu \hat{y}_j^{\tilde{b}}(0) = 0$ for all $|\mu| \leqslant k$ and $j = 2, \ldots, r$, using the relation between b and $\tilde{b}$, one can easily verify that $||a - \tilde{b}||_{(\ell_1(\mathbb{Z}^s))^{r\times r}} \leqslant C_2||a-b||_{(\ell_1(\mathbb{Z}^s))^{r\times r}}$ for some constant C_2 independent of b. Replace η by the smaller η'. By Lemma 3.3 and $\hat{y}_1^a(0) = \hat{y}_1^{\tilde{b}}(0) = \hat{y}_1^b(0) = 1$, $\mathscr{F}_{k,y^{\tilde{b}},p} = \mathscr{F}_{k,y^a,p}$. By what has been proved in (1), we have the following estimate:

$$||\phi^{\tilde{b}} - \phi^a||_{(W_p^k(\mathbb{R}^s))^{r\times 1}} \leqslant C_1||a - \tilde{b}||_{(\ell_1(\mathbb{Z}^s))^{r\times r}} \leqslant C_1C_2||a - b||_{(\ell_1(\mathbb{Z}^s))^{r\times r}}$$

and consequently, $||\phi^{\tilde{b}}||_{(W_p^k(\mathbb{R}^s))^{r\times 1}} \leqslant 2||\phi^a||_{(W_p^k(\mathbb{R}^s))^{r\times 1}}$ for small enough η. On the other hand, since $\hat{\phi}^b(\xi) - \hat{\phi}^{\tilde{b}}(\xi) = (\hat{U}(\xi) - I_r)\hat{\phi}^{\tilde{b}}(\xi)$, we deduce that

$$\begin{aligned} ||\phi^b - \phi^{\tilde{b}}||_{(W_p^k(\mathbb{R}^s))^{r\times 1}} &\leqslant ||U - I_r\delta||_{(\ell_1(\mathbb{Z}^s))^{r\times r}}||\phi^{\tilde{b}}||_{(W_p^k(\mathbb{R}^s))^{r\times 1}} \\ &\leqslant 2CC_0||\phi^a||_{(W_p^k(\mathbb{R}^s))^{r\times 1}}||a - b||_{(\ell_1(\mathbb{Z}^s))^{r\times r}}. \end{aligned}$$

Consequently, taking $C_3 := C_1C_2 + 2CC_0||\phi^a||_{(W_p^k(\mathbb{R}^s))^{r\times 1}}$, we have estimate in (6.6) since

$$\begin{aligned} ||\phi^b - \phi^a||_{(W_p^k(\mathbb{R}^s))^{r\times 1}} &\leqslant ||\phi^b - \phi^{\tilde{b}}||_{(W_p^k(\mathbb{R}^s))^{r\times 1}} + ||\phi^{\tilde{b}} - \phi^a||_{(W_p^k(\mathbb{R}^s))^{r\times 1}} \\ &\leqslant C_3||a - b||_{(\ell_1(\mathbb{Z}^s))^{r\times r}}. \end{aligned}$$

By Lemma 6.1, we have $\rho_k(\tilde{b}, M, p, y^{\tilde{b}}) < m^{1/p-1}\rho(M)^{-k}$ since $\mathscr{V}_{k,y^{\tilde{b}}} = \mathscr{V}_{k,y^a}$. By the relation between b and $\tilde{b}$, it is easy to verify that $\rho_k(b, M, p, y^b) = \rho_k(\tilde{b}, M, p, y^{\tilde{b}})$. Consequently, $\rho_k(b, M, p, y^b) < m^{1/p-1}\rho(M)^{-k}$. $\square$

The proofs of Lemma 6.1 and Theorem 6.2 yield that $\lim_{\eta\to 0, b\in N_\eta(a,k,M,\Omega)} v_p(b, M) = v_p(a, M)$ which is not a trivial fact since the space $\mathscr{V}_{k,y^b}$ in the definition of $v_p(b, M)$ changes with b and is not invariant under perturbation.

7. Computing the important quantity $v_2(a, M)$

Since the quantity $v_p(a, M)$, which is defined in (4.3), plays a very important role in characterizing the convergence of a vector cascade algorithm in a Sobolev space and in characterizing the L_p smoothness of a refinable function vector, it is of interest to find a numerical algorithm for efficiently computing or estimating the quantity $v_p(a, M)$.

For a matrix A, we denote $A^* := \bar{A}^T$. For $u, v \in (\ell_2(\mathbb{Z}^s))^{m\times n}$, define

$$\langle u, v\rangle := \mathrm{trace}\left(\sum_{\beta\in\mathbb{Z}^s} u(\beta)v(\beta)^*\right) = \mathrm{trace}\left(\frac{1}{(2\pi)^s}\int_{[-\pi,\pi)^s} \hat{u}(\xi)\hat{v}(\xi)^*\, d\xi\right). \quad (7.1)$$

There are two operators $\mathscr{S}_{a,M}$ and $\mathscr{T}_{a,M}$ which are convolved version of the operators $S_{a,M}$ and $T_{a,M}$ in Proposition 2.5. Define $\mathscr{S}_{a,M}$ and

$\mathscr{T}_{a,M}$ to be

$$\mathscr{S}_{a,M}v(\alpha) = |\det M| \sum_{\beta,\gamma\in\mathbb{Z}^s} a(M\gamma-\alpha+\beta)^* v(\gamma)a(\beta), \quad \alpha\in\mathbb{Z}^s, \quad v\in(\ell_0(\mathbb{Z}^s))^{r\times r},$$

$$\mathscr{T}_{a,M}v(\alpha) = |\det M| \sum_{\beta,\gamma\in\mathbb{Z}^s} a(M\alpha-\gamma+\beta)v(\gamma)a(\beta)^*, \quad \alpha\in\mathbb{Z}^s, \quad v\in(\ell_0(\mathbb{Z}^s))^{r\times r}.$$

It is easy to check that $\langle \mathscr{S}_{a,M}u, v\rangle = \langle u, \mathscr{T}_{a,M}v\rangle$ for all $u, v\in(\ell_0(\mathbb{Z}^s))^{r\times r}$. Recall that spec$(A)$ denotes the multiset of all the eigenvalues of A counting the multiplicity of the eigenvalues.

It is known in the literature that $v_2(a, M)$ can be computed by finding the spectral radius of a finite matrix (see [8,11,16,18,21,26,28,32,33,42,43] and references therein). In the vector case, Jia and Jiang [28] found the following algorithm for computing $v_2(a, M)$ for an isotropic dilation matrix M for which we shall provide a self-contained and simple proof here.

Theorem 7.1. *Let M be an $s\times s$ dilation matrix and $\sigma = (\sigma_1, \ldots, \sigma_s)$, where* $\operatorname{spec}(M) = \{\sigma_1, \ldots, \sigma_s\}$. *Let a be a finitely supported mask on $\mathbb{Z}^s$ with multiplicity r such that a satisfies the sum rules of the highest possible order $k+1$ but not $k+2$ in* (2.8) *and* (2.9) *with some sequence $y\in(\ell_0(\mathbb{Z}^s))^{1\times r}$ and $\hat{y}(0)\neq 0$. Then the quantity $v_2(a, M)$ defined in* (4.3) *can be calculated by the following procedure*:

(1) *Form a new sequence $b\in(\ell_0(\mathbb{Z}^s))^{r^2\times r^2}$ by*

$$b(\alpha) := |\det M| \sum_{\beta\in\mathbb{Z}^s} \overline{a(\beta)} \otimes a(\alpha+\beta), \quad \alpha\in\mathbb{Z}^s; \tag{7.2}$$

(2) *Calculate the set $K = \mathbb{Z}^s \cap \sum_{j=1}^{\infty} M^{-j}(\operatorname{supp} b)$, where $\operatorname{supp} b := \{\beta\in\mathbb{Z}^s\colon\ b(\beta)\neq 0\}$;*

(3) *Define the set E_k to be*

$$E_k := \{\lambda\overline{\sigma^{-\mu}}, \bar{\lambda}\sigma^{-\mu}\colon\ \lambda\in\operatorname{spec}(\hat{a}(0))\backslash\{1\},\ |\mu|\leqslant k\}$$
$$\cup\{\sigma^{-\mu}\colon\ |\mu|\leqslant 2k+1\}. \tag{7.3}$$

Then the quantity $\rho_k(a, M, p, y)$, which is defined in (4.1), *is given by $\sqrt{\rho_k/|\det M|}$, where*

$$\rho_k := \max\{|\lambda|\colon\ \lambda\in\operatorname{spec}((b(M\alpha-\beta))_{\alpha,\beta\in K})\backslash E_k\}.$$

Consequently, $v_2(a, M) = -\log_{\rho(M)} \sqrt{\rho_k}$.

Proof. By Proposition 2.4, without loss of generality, we can assume $\hat{y}(\xi) = [\hat{y}_1(\xi), 0, \ldots, 0]$. By assumption on mask a and Proposition 2.4, $\hat{a}(\xi)$ must take the form of (2.10) such that (2.11) holds. Let $W_k := \operatorname{span}\{w\colon\ \hat{w}(\xi) =$

$\hat{u}(\xi)\hat{v}(\xi)^*,\ u,v\in\mathscr{V}_{k,y}\}$. Since $\mathscr{V}_{k,y}=\mathscr{V}_{k,\delta}\times(\ell_0(\mathbb{Z}^s))^{(r-1)\times 1}$, we can easily deduce that

$$W_k=\left\{\begin{bmatrix} v_{1,1} & v_{1,2}\\ v_{2,1} & v_{2,2}\end{bmatrix}:\ v_{1,1}\in\mathscr{V}_{2k+1,\delta}, v_{1,2}\in(\mathscr{V}_{k,\delta})^{1\times(r-1)},\right.$$

$$\left. v_{2,1}\in(\mathscr{V}_{k,\delta})^{(r-1)\times 1}, v_{2,2}\in(\ell_0(\mathbb{Z}^s))^{(r-1)\times(r-1)}\right\}.$$

For a finite-dimensional subspace V of $(\ell_0(\mathbb{Z}^s))^{r\times r}$ such that $\mathscr{T}_{a,M}V\subseteq V$, it was proved in [21] that $\operatorname{spec}(\mathscr{T}_{a,M}|_V)\cup\{0\}=\operatorname{spec}(\mathscr{T}_{a,M}|_{V\cap(\ell(K))^{r\times r}})\cup\{0\}$. So, for simplicity, $\operatorname{spec}(\mathscr{T}_{a,M}|_V)$ always means $\operatorname{spec}(\mathscr{T}_{a,M}|_{V\cap(\ell(K))^{r\times r}})$.

Let $m:=|\det M|$. For $v\in\mathscr{V}_{k,y}$, let w denote the sequence given by $\hat{w}(\xi)=\hat{v}(\xi)\hat{v}(\xi)^*$. By induction, one has

$$\begin{aligned}(2\pi)^s\|a_n*v\|^2_{(\ell_2(\mathbb{Z}^s))^{r\times 1}} &=(2\pi)^s\langle a_n*v,a_n*v\rangle\\ &=\operatorname{trace}\left(\int_{[-\pi,\pi)^s}\hat{a}_n(\xi)\hat{v}(\xi)\hat{v}(\xi)^*\hat{a}_n(\xi)^*\,d\xi\right)\\ &=\frac{1}{m^n}\operatorname{trace}\left(\int_{[-\pi,\pi)^s}\widehat{\mathscr{T}^n_{a,M}w}(\xi)\,d\xi\right).\end{aligned}$$

For $w\in(\ell_0(\mathbb{Z}^s))^{r\times r}$ such that $\hat{w}(\xi)\geqslant 0$ (that is, $\hat{w}(\xi)$ is positive semidefinite), we have $\mathscr{T}^n_{a,M}w(\xi)\geqslant 0$ and

$$\begin{aligned}\operatorname{trace}\left(\int_{[-\pi,\pi)^s}\widehat{\mathscr{T}^n_{a,M}w}(\xi)\,d\xi\right) &\leqslant\int_{[-\pi,\pi)^s}\|\widehat{\mathscr{T}^n_{a,M}w}(\xi)\|_{\ell_1}\,d\xi\\ &\leqslant r\times\operatorname{trace}\left(\int_{[-\pi,\pi)^s}\widehat{\mathscr{T}^n_{a,M}w}(\xi)\,d\xi\right).\end{aligned}$$

By the Cauchy–Schwartz inequality, we deduce that

$$\rho_k(a,M,2,y)=\sup\{\|a_n*v\|^{1/n}_{(\ell_2(\mathbb{Z}^s))^{r\times 1}}:\ v\in\mathscr{V}_{k,y}\}=\sqrt{\rho(\mathscr{T}_{a,M}|_{W_k})/m}.$$

In order to calculate $\rho(\mathscr{T}_{a,M}|_{W_k})$, we define three types of subspaces U^1_j,U^2_j,U^3_j of $(\Pi_j)^{r\times r}$ by

$$U^1_j:=\left\{\begin{bmatrix} p & 0\\ 0 & 0\end{bmatrix}:\ p\in\Pi_j\right\},\quad U^2_j:=\left\{\begin{bmatrix} 0 & 0\\ p & 0\end{bmatrix}:\ p\in(\Pi_j)^{(r-1)\times 1}\right\},\quad j\in\mathbb{N}_0 \tag{7.4}$$

and $U_j^3 := \{p\colon p^T \in U_j^2\}$. Due to the special form of $\hat{a}(\xi)$ in (2.10) and (2.11), by a simple computation, it follows directly from (2.19) and (2.21) that

$$\mathscr{S}_{a,M}\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m \sum_{\beta,\gamma\in\mathbb{Z}^s} a_{1,1}^*(M\gamma - \cdot + \beta)p(\gamma)a_{1,1}(\beta) & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} S_{c,M}p & 0 \\ 0 & 0 \end{bmatrix}, \quad p\in\Pi_{2k+1},$$

where $c\in\ell_0(\mathbb{Z}^s)$ is given by $c(\alpha) = \sum_{\beta\in\mathbb{Z}^s} \overline{a_{1,1}(\beta-\alpha)}a_{1,1}(\beta)$; that is, $\hat{c}(\xi) = |\hat{a}_{1,1}(\xi)|^2$. Since $a_{1,1}$ satisfies the sum rules of order $k+1$, so c satisfies the sum rules of order $2k+2$. By (6) in Proposition 2.4, $S_{c,M}p - p(M^{-1}\cdot)\in\Pi_{\deg(p)-1}$ for all $p\in\Pi_{2k+1}$. Therefore, $S_{c,M}p \equiv p(M^{-1}\cdot) \bmod \Pi_{j-1}$ for all $p\in\Pi_j/\Pi_{j-1}$ and $j=0,\dots,2k+1$. Consequently,

$$\operatorname{spec}(S_{c,M}|_{\Pi_j/\Pi_{j-1}}) = \operatorname{spec}(S(M^{-1},O_j)) = \{\sigma^{-\mu}\colon |\mu| = j\},$$

where $S(M^{-1},O_j)$ is defined in (2.1). Hence,

$$\operatorname{spec}(\mathscr{S}_{a,M}|_{U^1_{2k+1}}) = \operatorname{spec}(S_{c,M}|_{\Pi_{2k+1}}) = \bigcup_{j=0}^{2k+1} \operatorname{spec}(S_{c,M}|_{\Pi_j/\Pi_{j-1}})$$
$$= \{\sigma^{-\mu}\colon |\mu|\leqslant 2k+1\}.$$

By Proposition 2.4 and a simple computation, it follows from (2.19) and (2.21) that for all $p\in(\Pi_k)^{(r-1)\times 1}$,

$$\mathscr{S}_{a,M}\begin{bmatrix} 0 & 0 \\ p & 0 \end{bmatrix} = m\begin{bmatrix} \sum_{\beta,\gamma\in\mathbb{Z}^s} a_{2,1}^*(\beta)p(\gamma)a_{1,1}(\cdot - M\gamma+\beta) & 0 \\ \sum_{\beta,\gamma\in\mathbb{Z}^s} a_{2,2}^*(\beta)p(\gamma)a_{1,1}(\cdot - M\gamma+\beta) & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{\beta\in\mathbb{Z}^s} a_{2,1}^*(\beta)[S_{a_{1,1},M}p](\cdot+\beta) & 0 \\ \sum_{\beta\in\mathbb{Z}^s} a_{2,2}^*(\beta)[S_{a_{1,1},M}p](\cdot+\beta) & 0 \end{bmatrix}. \tag{7.5}$$

Since $a_{1,1}$ satisfies the sum rules of order of $k+1$, by Proposition 2.5 and $\hat{a}_{2,2}(0) = \sum_{\beta\in\mathbb{Z}^s} a_{2,2}(\beta)$, for $p\in(\Pi_k)^{(r-1)\times 1}$, we have

$$\sum_{\beta\in\mathbb{Z}^s} a_{2,2}^*(\beta)[S_{a_{1,1},M}p](\cdot+\beta) \equiv \sum_{\beta\in\mathbb{Z}^s} a_{2,2}^*(\beta)p(M^{-1}(\cdot+\beta))$$
$$\equiv \hat{a}_{2,2}(0)^* p(M^{-1}\cdot) \quad \bmod\ (\Pi_{\deg(p)-1})^{(r-1)\times 1}.$$

Consequently, the quotient group $(U_j^2 \oplus U_{2k+1}^1)/(U_{j-1}^2 \oplus U_{2k+1}^1)$ is invariant under $\mathscr{S}_{a,M}$ and for $j = 0, \ldots, k$,

$$\mathscr{S}_{a,M}\begin{bmatrix} 0 & 0 \\ p & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 \\ \hat{a}_{2,2}(0)^* p(M^{-1}\cdot) & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 \\ p & 0 \end{bmatrix} \in (U_j^2 \oplus U_{2k+1}^1)/(U_{j-1}^2 \oplus U_{2k+1}^1).$$

Now by $\operatorname{spec}(\hat{a}_{2,2}(0)) = \operatorname{spec}(\hat{a}(0))\backslash\{1\}$, it is easy to verify that

$$\begin{aligned}
&\operatorname{spec}(\mathscr{S}_{a,M}|_{(U_j^2 \oplus U_{2k+1}^1)/(U_{j-1}^2 \oplus U_{2k+1}^1)}) \\
&\quad = \{\lambda\eta\colon\ \lambda \in \operatorname{spec}(\hat{a}_{2,2}(0)^*), \eta \in \operatorname{spec}(S(M^{-1}, O_j))\} \\
&\quad = \{\bar{\lambda}\sigma^{-\mu}\colon\ \lambda \in \operatorname{spec}(\hat{a}(0))\backslash\{1\}, |\mu| = j\}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\operatorname{spec}(\mathscr{S}_{a,M}|_{(U_k^2 \oplus U_{2k+1}^1)/U_{2k+1}^1}) &= \bigcup_{j=0}^{k} \operatorname{spec}(\mathscr{S}_{a,M}|_{(U_j^2 \oplus U_{2k+1}^1)/(U_{j-1}^2 \oplus U_{2k+1}^1)}) \\
&= \{\bar{\lambda}\sigma^{-\mu}\colon\ \lambda \in \operatorname{spec}(\hat{a}(0))\backslash\{1\}, |\mu| \leqslant k\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\operatorname{spec}(\mathscr{S}_{a,M}|_{(U_k^3 \oplus U_{2k+1}^1)/U_{2k+1}^1}) &= \bigcup_{j=0}^{k} \operatorname{spec}(\mathscr{S}_{a,M}|_{(U_j^3 \oplus U_{2k+1}^1)/(U_{j-1}^3 \oplus U_{2k+1}^1)}) \\
&= \{\lambda\overline{\sigma^{-\mu}}\colon\ \lambda \in \operatorname{spec}(\hat{a}(0))\backslash\{1\}, |\mu| \leqslant k\}.
\end{aligned}$$

By the definition of $\mathscr{V}_{k,\delta}$, $\mathscr{V}_{k,\delta} = \Pi_k^{\perp}$. It is straightforward to see that $W_k = (U_{2k+1}^1 \oplus U_k^2 \oplus U_k^3)^{\perp}$. By the duality relation, we conclude that

$$\begin{aligned}
\operatorname{spec}(\mathscr{T}_{a,M}|_{(\ell_0(\mathbb{Z}^s))^{r\times r}/W_k}) &= \operatorname{spec}(\mathscr{S}_{a,M}|_{U_{2k+1}^1 \oplus U_k^2 \oplus U_k^3}) \\
&= \operatorname{spec}(\mathscr{S}_{a,M}|_{U_{2k+1}^1}) \cup \operatorname{spec}(\mathscr{S}_{a,M}|_{U_k^2 \oplus U_{2k+1}^1/U_{2k+1}^1}) \\
&\quad \cup \operatorname{spec}(\mathscr{S}_{a,M}|_{U_k^3 \oplus U_{2k+1}^1/U_{2k+1}^1}) \\
&= E_k.
\end{aligned}$$

Using the vec operation as discussed in Lemma 2.2, it is easy to see that $\operatorname{spec}(\mathscr{T}_{a,M}|_{(\ell(K))^{r\times r}}) = \operatorname{spec}((b(M\alpha - \beta))_{\alpha,\beta\in K})$. Therefore, $\operatorname{spec}(\mathscr{T}_{a,M}|_{W_k}) = \operatorname{spec}((b(M\alpha - \beta))_{\alpha,\beta\in K})\backslash E_k$ which completes the proof. $\square$

The above proof can be carried out similarly by using $\mathcal{T}_{a,M}$ directly instead of using $\mathcal{S}_{a,M}$ (see [19]). The above proof can be also easily adapted to take into account the symmetry of the mask. For computing $v_2(a,M)$ for scalar masks by taking into account symmetry to significantly reduce the size of the problem, see [19]. One way of computing the set K in Theorem 7.1 is as follows. Choose any initial finite subset K_0 of $\mathbb{Z}^s$ such that $K \subseteq K_0 \subseteq \mathbb{Z}^s$. Recursively define $K_j := K_{j-1} \cap M^{-1}(K_{j-1} + \operatorname{supp} b), j \in \mathbb{N}$. Then there must exist some j such that $K_j = K_{j-1}$. An easy argument shows that $K = K_j$. For more detail, see [19, Proposition 2.2]. From the proof of Theorem 7.1, we observe that K can be replaced by any finite subset K_0 of $\mathbb{Z}^s$ such that $M^{-1}(K_0 + \operatorname{supp} b) \cap \mathbb{Z}^s \subseteq K_0$ and for every $0 \leqslant j \leqslant k$, there is a subset B_j of $(\ell(K_0))^{r\times 1}$ such that B_j generates $\mathcal{V}_{j,y}$; that is, $\operatorname{span}\{v(\cdot - \beta)\colon v \in B_j, \beta \in \mathbb{Z}^s\} = \mathcal{V}_{j,y}$.

In the univariate case, one can compute $v_2(a,M)$ by factorizing the symbol of a mask [7,38,41] as follows.

Proposition 7.2. *Let M be an integer such that $|M| > 1$. Let a be a matrix mask on $\mathbb{Z}$ such that a satisfies the sum rules of order $k+1$ in* (2.8) *and* (2.9) *with some sequence $y \in (\ell_0(\mathbb{Z}))^{1\times r}$. Let U_y be given in Proposition* 2.4 *so that $\hat{U}_y(M\xi)^{-1}\hat{a}(\xi)\hat{U}_y(\xi)$ takes the form of* (2.10). *Define a new sequence b by*

$$\hat{b}(\xi) := \begin{bmatrix} (1-e^{-iM\xi})^{k+1} & 0 \\ 0 & I_{r-1} \end{bmatrix}^{-1} \times \hat{U}_y(M\xi)^{-1}\hat{a}(\xi)\hat{U}_y(\xi) \begin{bmatrix} (1-e^{-i\xi})^{k+1} & 0 \\ 0 & I_{r-1} \end{bmatrix}. \tag{7.6}$$

Then b is a finitely supported sequence on $\mathbb{Z}$ and

$$\rho_k(a,M,p,y) = \rho_{-1}(b,M,p,0) := \lim_{n\to\infty} \|b_n\|^{1/n}_{(\ell_p(\mathbb{Z}))^{r\times r}},$$

where $\hat{b}_n(\xi) = \hat{b}(M^{n-1}\xi)\cdots\hat{b}(M\xi)\hat{b}(\xi)$. Moreover, $\rho_k(a,M,2,y) = \sqrt{\rho_k/|M|}$, where ρ_k is the spectral radius of $(\sum_{\beta\in\mathbb{Z}} \overline{b(\beta)} \otimes b(\alpha+\beta))_{\alpha,\beta\in K}$, where $K := \mathbb{Z} \cap \sum_{j=1}^{\infty} M^{-j}(\operatorname{supp} a - \operatorname{supp} a)$.

Proof. By Proposition 2.4, it suffices to prove it for the case $\hat{y}(\xi) = [\hat{y}_1(\xi), 0, \ldots, 0]$ and $\hat{U}_y(\xi) = I_r$. By (2.10) and (2.11), b must be a finitely supported sequence. Let w be given by $\hat{w}(\xi) = \operatorname{diag}((1-e^{-i\xi})^{k+1}, I_{r-1})$. We observe that $\{we_j\colon j = 1, \ldots, r\} = \mathcal{B}_{k,y}$ generates $\mathcal{V}_{k,y}$ and $\hat{a}(\xi) = \hat{w}(M\xi)\hat{b}(\xi)\hat{w}(\xi)^{-1}$. Consequently, we have

$$\rho_k(a,M,p,y) = \lim_{n\to\infty} \|a_n * w\|^{1/n}_{(\ell_p(\mathbb{Z}))^{r\times r}}$$

and

$$\widehat{a_n * w}(\xi) = \hat{a}_n(\xi)\hat{w}(\xi) = \hat{w}(M^n\xi)\hat{b}_n(\xi), \quad n \in \mathbb{N}.$$

Since all a, b and w are finitely supported, we assume that they are supported on $[-N, N]$ for some positive integer N. Define

$$\hat{c}_n(\xi) := \sum_{j=0}^{3N-1} \operatorname{diag}(e^{-iM^n j\xi}, I_{r-1}) \widehat{a_n * w}(\xi).$$

Note that

$$\hat{b}_n(\xi) = \hat{w}(M^n\xi)^{-1}\widehat{a_n * w}(\xi) = \sum_{j=0}^{\infty} \operatorname{diag}(e^{-iM^n j\xi}, I_{r-1}) \widehat{a_n * w}(\xi).$$

It is easy to see that b_n vanishes outside $[-M^nN, M^nN]$ and $b_n(\beta) = c_n(\beta)$ for all $\beta \in \mathbb{Z} \cap [-M^nN, M^nN]$. Therefore, we have

$$\begin{aligned}(3N)^{-1}\|b_n\|_{(\ell_p(\mathbb{Z}))^{r\times r}} &\leqslant (3N)^{-1}\|c_n\|_{(\ell_p(\mathbb{Z}))^{r\times r}} \\ &\leqslant \|a_n * w\|_{(\ell_p(\mathbb{Z}))^{r\times r}} \leqslant \|w\|_{(\ell_1(\mathbb{Z}))^{r\times r}}\|b_n\|_{(\ell_p(\mathbb{Z}))^{r\times r}}.\end{aligned}$$

Consequently,

$$\rho_k(a, M, p, y) = \lim_{n\to\infty} \|a_n * w\|^{1/n}_{(\ell_p(\mathbb{Z}))^{r\times r}} = \lim_{n\to\infty} \|b_n\|^{1/n}_{(\ell_p(\mathbb{Z}))^{r\times r}}$$

which completes the proof. $\square$

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Note added in the revised version. It is straightforward to see that all the results and proofs in the paper hold for a general (not necessarily isotropic) dilation matrix when $k = 0$. After submitting this paper, we became aware that the convergence of a vector cascade algorithm in $L_p(\mathbb{R}^s)$ with a general dilation matrix has also been obtained in Li [35,36] (that is, the equivalence between (1) and (10) for the case $k = 0$ in Theorem 4.3).

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